NON-LINEAR VIBRATIONS OF HYPERELASTIC CIRCULAR MEMBRANES WITH CONTINUOUSLY VARYING THICKNESS

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Abstract. Although dozens of theoretical studies have been published on linear vibrations of membranes with continuously varying density, little is known in the literature on the linear and nonlinear vibrations of hyperelastic membranes with varying thickness. So, the aim of the present work is to investigate the nonlinear vibration response of a pre-stretched circular hyperelastic membrane with varying thickness subjected to finite deformations. The membrane is composed of an isotropic, homogeneous, incompressible and hyperelastic material, which is modeled a neo-Hookean material. Initially the density of membrane is considered variable and later the thickness is considered variable, both along your radial direction. First the solution of the membrane under a uniform radial stretch is obtained. Then the equations of motion of the pre-stretched membrane are derived. From the linearized equations, the natural frequencies and mode shapes of the membrane are obtained analytically. The solutions are obtained in terms of hypergeometric functions. Then the natural modes are used to approximate the nonlinear deformation field using the Galerkin method. The results are compared with the results evaluated for the same membrane using a nonlinear finite element formulation. Excellent agreement is observed up to very large deflections. The results show the strong influence of the stretching ratio on the linear and nonlinear oscillations of the membrane.

Keywords: circular membranes, hyperelastic material, non-linear vibrations, varying thickness.

1. INTRODUCTION

Membranes play a significant role in nature due its high load-carrying capacity per unit weight. The analysis of membrane mechanics is an important topic in nonlinear continuum mechanics. In particular the study of hyperelastic membranes under finite deformations, such as elastomeric membranes (Gonçalves et al., 2003), is a rather challenging subject and, in such cases, elasticity in the fully non-linear range must be employed. The pioneering works of Rivlin (1948) and Green and Adkins (1960) on non-linear elasticity set up the basis for the analysis of structures under large deformations. Strain-invariant constitutive models are usually used to describe the behavior of hyperelastic materials, one of the simplest constitutive model is the neo-Hookean model.

Several analytical and numerical analyses are found in literature on the linear vibrations of membranes with continuously varying density and different geometries, for example Laura et al (1998), Gutierrez et al (1998), Wang (1998) and Buchanan (2005). Regarding the influence of varying thickness on the free vibration characteristics of the membrane, little is known in literature. However, both variations are found in practical applications and particularly in nature. Circular and annular membranes with a polynomial variation of the density in the radial direction are studied by Jabareen and Eisenberger (2001). The natural frequencies and mode shapes are obtained by the method of Frobenius (Humi and Miller, 1998). Using also Frobenius method, Willatzen (2002) presents a semi-analytical approximation for the axi-symmetric modes shapes and related natural frequencies for circular and annular membranes with continuously varying density in the radial direction. Several density distributions are considered in their analysis. Subrahmanyam and Sujith (2001) obtain an analytical solution for the axi-symmetric vibrations of circular and annular membranes with varying density in the radial direction. Through a change of variables, the linear equation of motion of the membrane is transformed into either a Kummer’s confluent hypergeometric differential equation whose solution can be expressed in terms of Kummer’s confluent hypergeometric function or a Bessel functions (Humi and Miller, 1998). A review of the literature on the static and dynamic behavior of membranes, both theoretical and experimental, can be found in Jenkins and Leonard (1991), Jenkins (1996) and Jenkins and Korde (2006).

So, the aim of the present work is to study the linear and non-linear free vibrations of a pre-stretched circular hyperelastic membrane with varying density or thickness. The membrane material is assumed to be isotropic and incompressible and its behavior is described by the neo-Hookean constitutive law. A variational formulation, considering finite deformations, is used to derive the equilibrium equations of the membrane under a uniform radial stretch and the equations of motion of the pre-stretched membrane. The linear and non-linear vibrations are analyzed and the influence of the pre-stretch on these results is evaluated. The problem is also analyzed using the finite element software Abaqus 6.5.

2. PROBLEM FORMULATION
First consider a circular hyperelastic membrane of undeformed external radius $R_0$, thickness $h$ and mass density $\Gamma$. The membrane is uniformly stretched in radial direction, reaching a final radius $R_f$, and then fixed along this edge. After the application of the static radial traction, the membrane is perturbed in the transversal direction. The deformed and undeformed geometries, co-ordinate system are shown in Fig. 1.

Figure 1 – Deformed and undeformed configurations of the stretched membrane in the radial direction

The strain energy density can be written as a function of the principal stretches $\lambda_1$, $\lambda_2$ and $\lambda_3$ or, alternatively, of the strain invariants $I_1$, $I_2$ and $I_3$ (Green and Adkins, 1960). The three strain invariants of the deformation field can be written in terms of the principal stretches $\lambda_i$ (i=1, 2, 3) as:

$$I_1 = \lambda_1^2 + \lambda_2^2 + \lambda_3^2; \quad I_2 = (\lambda_1\lambda_2)^2 + (\lambda_2\lambda_3)^2 + (\lambda_1\lambda_3)^2; \quad I_3 = (\lambda_1\lambda_2\lambda_3)^2$$

(1)

The membrane material is homogeneous, isotropic and incompressible ($I_3 = 1$). Considering a neo-Hookean material, the energy density function can be described as:

$$W = C_1(I_1 - 3)$$

(2)

where $C_1$ is a material parameter and $I_1$ is the first strain invariant.

2.1. Static Analysis

For a uniformly stretched membrane in the radial direction, the radial displacement function - $r_\rho(\rho)$ - must satisfy the following non-linear differential equilibrium equation:

$$\frac{r_\rho}{\rho} - \frac{3\rho^3}{r_\rho^3} \frac{3\rho^2}{r_\rho^2} + \frac{3\rho}{r_\rho} \frac{1}{r_\rho} \frac{\rho}{r_\rho} \frac{r_\rho^2}{r_\rho^2} \frac{r_\rho}{r_\rho} \frac{1}{r_\rho} = 0$$

(3)

and the following boundary conditions:

$$r_\rho(R_0) = R_f$$

(4)

where $\frac{\rho}{\rho} \frac{\rho}{\rho} \frac{\rho}{\rho} \frac{\rho}{\rho}$. The exact solution of equation (3), satisfying the boundary condition (4), for the circular membrane with variable density is given by

$$r_\rho(\rho) = \delta \rho$$

(5)

where $\delta = R_f / R_0$ is the radial stretch ratio.

The solution of Eq. (3) for the circular membrane with variable thickness must be obtained by approximation techniques. Here the solution is obtained by the shooting method (Press et al., 2007). The differential equation of the boundary value problem is integrated using the Runge-Kutta integration scheme and the free boundary conditions at the initial point are adjusted by the Newton-Raphson method (Soares, 2009). Based on the results obtained by the shooting method, the radius of the extended circular membrane can be described by the following function of $\rho$:

$$r_\rho(\rho) = a_1 \rho^4 + a_2 \rho^3 + a_3 \rho^2 + a_4 \rho$$

(6)
where \( a_i \) are constants that depend on the properties of the deformed membrane. The static transversal and circumferential displacement components, \( z_\theta \) and \( \beta_\theta \), are zero.

### 2.2. Dynamic Analyses

Now the stretched membrane is perturbed in the transversal direction and its non-linear vibrations are analyzed. So the co-ordinates of a deformed point are considered as the sum of two parts, that is:

\[
\begin{align*}
  r(\rho, \theta, t) &= r_0(\rho, \theta) + u(\rho, \theta, t) \\
  \beta(\rho, \theta, t) &= \theta + \beta_0(\rho, \theta) + v(\rho, \theta, t) \\
  z(\rho, \theta, t) &= z_0(\rho, \theta) + w(\rho, \theta, t)
\end{align*}
\]

(7)

where \( u(\rho, \theta, t) \), \( v(\rho, \theta, t) \) and \( w(\rho, \theta, t) \) are the perturbation components in the radial, circumferential and transversal directions, respectively and \( r_0(\rho, \theta), \beta_0(\rho, \theta) \) and \( z_0(\rho, \theta) \) describes the initial deformed static state.

Through the results of the FE analysis of this problem, one can observed that, during the transversal non-linear vibrations of the pre-stretched membrane, the in-plane displacements \( u \) and \( v \) are negligible compared with the transversal displacement \( w \) (Soares, 2009).

### 2.3. Variable Density

First, the influence of the continuously varying density is analyzed. In this case, the membrane thickness, \( h_m \), is constant and the material density varies in the radial direction according to the law (Jabareen e Eisenberger, 2001; Subrahmanyam e Sujith, 2001; Willatzen, 2002):

\[
\Gamma(\rho) = \Gamma_o (1 + \kappa \rho^2)
\]

(8)

where \( \Gamma(\rho) \) is the material density at a point with radial coordinate \( \rho \), \( \Gamma_o \) is a constant and the parameter \( \kappa \) describes the variation of the material mass density along the undeformed radius of the membrane. The linear equation of motion of the membrane with varying density is obtained by applying Hamilton’s principle and is given by (Soares, 2009):

\[
\frac{2C_1}{\Gamma_o(1 + \kappa \rho^2)} \left(1 - \frac{1}{\delta^6} \frac{\partial^2 w}{\partial \rho^2} + \frac{1}{\rho \delta^2} \frac{\partial w}{\partial \rho} + \frac{1}{\rho^2 \delta^2} \frac{\partial^2 w}{\partial \theta^2} \right) \frac{\partial^2 w}{\partial t^2} = 0
\]

(9)

The transversal displacement \( w \) is obtained by transforming the linear equation of motion (9) into an equation that has a known analytical solution. Solving the partial differential equation by separation of variables \((\rho, \theta, t)\), the transversal displacement can be written as:

\[
w(\rho, \theta, t) = A_{mn} G(\rho) \cos(n \theta) \cos(\omega_{mn} t)
\]

(10)

where \( A_{mn} \) is the modal amplitude; \( G(\rho) \), is an unknown function of \( \rho \), \( m \), is the number of half-waves in the radial direction; \( n \), the number of circumferential waves and \( \omega_{mn} \), the free vibration frequency associated to mode \((m, n)\).

Substituting (10) into (9), the following linear differential equation in \( \rho \) with variables coefficients, known as Whittaker differential equation (Abramowitz e Stegun, 1972), is obtained:

\[
\frac{d^2 G(\rho)}{d \rho^2} + \frac{1}{\rho} \frac{d G(\rho)}{d \rho} + \left( \frac{\Gamma_o(1 + \kappa \rho^2) \delta^6 \omega_{mn}^2}{2C_1(\delta^6 - 1)} - \frac{n^2}{\rho^2} \right) G(\rho) = 0
\]

(11)

Solving equation (11) together with the boundary conditions in the radial direction (Abramowitz e Stegun, 1972), the following expression for the membrane vibration modes is obtained:

\[
w(\rho, \theta, t) = A_{mn} M_n \left( \frac{1}{4} \sqrt{\frac{\Gamma_o k_{mn} \delta^6}{2C_1(\delta^6 - 1)\kappa^2}} \right) \frac{n}{2} \left( \frac{\Gamma_o k_{mn} \delta^6}{2C_1(\delta^6 - 1)\kappa^2} \right)^{1/2} \frac{\rho}{\kappa \rho^2} \frac{\rho}{\kappa \rho^2}^2 \frac{1}{\rho} \cos(n \theta) \cos(\omega_{mn} t)
\]

(12)

and \( M_n \) is Whittaker’s function of order \( n \) (Abramowitz e Stegun, 1972); \( k_{mn} \), is the \( m \)-th root of \( Z(k_{mn}) \). By substituting the transversal displacement (12) into Eq. (11), the natural frequencies of the membrane are obtained:

\[
\omega_{mn} = \sqrt{k_{mn}}
\]

(13)
The nonlinear equation of motion of the membrane in the transversal direction is given by

\[-\frac{\partial}{\partial \rho} \left( \rho \frac{\partial W}{\partial z, \rho} \right) - \frac{\partial}{\partial \theta} \left( \rho \frac{\partial W}{\partial z, \theta} \right) + \rho \Gamma_o (1 + \kappa \rho^2) \frac{\partial^2 W}{\partial t^2} = 0 \]  \hspace{1cm} (14)

The displacement field is then approximated by a sum of \(MnN\) linear vibration modes, Eq. (12), and equation (14) is solved by the Galerkin method.

2.4. Thickness Variation

For the circular membrane with variable thickness, the following variation law in the radial direction is considered:

\[ h(\rho) = h_o e^{n \rho^2} \]  \hspace{1cm} (15)

where \(h_o\) is a reference value and \(n\) is a constant that describes the variation of the thickness along the radial direction.

The membrane material density is constant and equal to \(\rho_0\). The linearized equation of motion of the hyperelastic membrane with continuously variable thickness in the radial direction is given as:

\[ 2C \left[ \frac{\rho^2}{r_o^4 r_o^4} - \frac{\partial^2 W^2}{\partial \rho^2} \right] + \frac{\partial^2 W^2}{\partial \theta^2} + \frac{3 \rho}{r_o^5 r_o^5} - 2 \rho^2 \frac{1}{r_o^5 r_o^5} - 4 \rho^2 \frac{1}{r_o^5 r_o^5} - 2 \rho^2 \frac{1}{r_o^5 r_o^5} - 2 \rho^2 \frac{1}{r_o^5 r_o^5} - 2 \rho^2 \frac{1}{r_o^5} + \frac{1}{\rho^2} \frac{\partial W}{\partial \rho} + \frac{1}{\rho^2} \frac{\partial^2 W}{\partial t^2} = 0 \]  \hspace{1cm} (16)

Solving the partial differential equation by separation of variables, the solution for the transversal displacement that satisfies the boundary conditions can be approximate as:

\[ w(\rho, \theta, t) = A_{mn} M_{m} \left( \omega_m, \Gamma_{o} \right) \cos(n \theta) \cos(\omega_m t) \]  \hspace{1cm} (17)

\[ B = \frac{R_f}{2 C R_o^{3/2} \left( r_o(R_o) \right)^3 - R_o^2} \]  \hspace{1cm} (18)

where \(b_{mn}\) is the \(m\)-th root of \(Z(\rho_{mn})\) and \(r_o'(R_o) = dr_o/R_o\). Substituting (17) into Error! Fonte de referência não encontrada. and applying the Galerkin method, the natural frequencies are obtained.

The nonlinear equation of motion in the transversal direction is given by:

\[-2\eta \rho \frac{\partial W}{\partial z, \rho} + \frac{\partial}{\partial \theta} \left( \rho \frac{\partial W}{\partial z, \theta} \right) + \rho \Gamma_o \frac{\partial^2 W}{\partial t^2} = 0 \]  \hspace{1cm} (19)

Again, the transversal displacement can be approximated by a sum of \(MnN\) linear modes and by applying the Galerkin-Urabe method. Through this procedure the nonlinear frequency-amplitude relation can be obtained.

3. NUMERICAL RESULTS

For the numerical analysis, a circular membrane with undeformed radius \(R_o = 1\ m\), initial thickness \(h_o = 0.001\ m\) and initial mass density \(\Gamma_o = 2200\ Kg/m^3\) is considered. The constant of the neo-Hookean, material is taken as \(C_i = 0.17\ MPa\) (Selvadurai, 2006). This membrane is also analyzed using the finite element software Abaqus 6.5, using the 2880 membrane elements M3D4R and M3D3 for the circular membrane with variable density and 1440 solid elements C3D8RH e C3DH for the circular membrane with variable thickness.

The vibration modes and frequencies are computed for increasing values of \(\kappa\) and \(\eta\). The analytical (AN) and finite element (FEM) results for the natural frequencies are compared in Tab. 1. The shape of the first three vibration modes is illustrated in Fig. 2. Figure 3 illustrates the variation of the lowest natural frequency of the membrane as a function of the stretching ratio \(\delta\) for different density (Fig. 3.a) and thickness (Fig. 3.b) variations. In all cases, the frequency increases as \(\delta\) increases and tends to a constant value for large values of \(\delta\). Also, for a given value of \(\delta\), the frequency increases as \(\kappa\) increases and as \(\eta\) decreases.

Figure 2 – Selected vibration modes of the circular membrane.
Table 1 – Vibration Frequencies (rad/s).

<table>
<thead>
<tr>
<th></th>
<th>Variable Density (δ = 1.1)</th>
<th>Variable Thickness (δ = 1.1)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>κ = −0.5</td>
<td>κ = 0.0</td>
</tr>
<tr>
<td>m, n</td>
<td>AN</td>
<td>FEM</td>
</tr>
<tr>
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<td>34.324</td>
<td>34.411</td>
</tr>
<tr>
<td>1, 2</td>
<td>42.773</td>
<td>42.581</td>
</tr>
</tbody>
</table>

Figure 3 – Lowest natural frequency, $\omega_{10}$, as a function of the stretching ratio, $\delta$.

The normalized frequency-amplitude relation for different values of $\kappa$ and $\eta$, associated with the lowest natural frequency that corresponds to the first axi-symmetric mode ($m = 1$ and $n = 0$), is shown in Fig. 4, considering for reference a point with coordinates (0, 0.5) in the membrane configuration and $\delta = 1.1$. The frequency-amplitude response is also evaluated from the free lightly damped response of the membrane, discretized by the FE method, using the methodology proposed by Nandakumar e Chatterjee (2005). The analytical results are favorably compared with the FE results in Fig. 5 for a membrane with $\delta = 1.10$, considering either variable density (Fig. 5(a)) or thickness (Fig. 5(b)).

Figure 4 – Normalized frequency-amplitude relation ($\delta = 1.1$).

Figure 5 – Frequency – transversal displacement for membrane with variable density and thickness ($(\rho, \theta) = (0, 0.5)$).
4. CONCLUSIONS

The mathematical modeling for the nonlinear vibration analysis of a pre-stretched circular hyperelastic membrane with varying inertial is presented in this paper. The membrane material is considered as homogeneous, isotropic and neo-Hookean. First the solution of the stretched membrane is obtained, showing that all relevant quantities are a function of the material constant and the stretching ratio only. Then, the equations of motion of the stretched circular membrane are obtained. By solving the linearized equations of motion, the vibration modes and frequencies of the hyperelastic membrane are obtained and these normal modes are used, together with the Galerkin method, to obtain an approximation of the nonlinear dynamic response. The same problem is also analyzed using the finite element software Abaqus. The results show the strong influence of the radial stretching ratio on the natural frequency and on the nonlinear frequency-amplitude relation. Also the variation of the density or thickness along the radial direction is evaluated. The good correlation between the non-linear FE results and those obtained in the present analysis corroborate the quality of the proposed reduced order model.

5. ACKNOWLEDGEMENTS

The authors acknowledge the financial support of the Brazilian research agencies CAPES, CNPq and FAPERJ.

6. REFERENCES


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