ON THE SPECTRAL PROPERTIES OF THE DOUBLE LAYER POTENTIAL MATRIX H OF THE BOUNDARY ELEMENT METHODS

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Abstract. The double-layer potential matrix $H$ of the conventional, collocation boundary element method (CBEM) is singular, as referred to a static problem in a bounded continuum. In elasticity, the singularity of $H$ means that no balanced forces can be related to rigid body displacements, which always have a simple representation and are spanned by the null space $N(H)$. On the other hand, the properties of $N(H^T)$, usually not relevant in the CBEM but of paramount importance in the variational implementations, may become quite intricate, as they are intimately related to concavities (notches, cracks, holes) of the discretized domain as well as to material non-homogeneities, as a result that local stress gradients can be represented by fundamental solutions only in a global sense. Symmetries and antisymmetries are also reflected in $N(H^T)$. The properties of $H$ and $H^T$ are investigated, also showing that in the usual implementations of the CBEM with real fundamental solutions all eigenvalues $\lambda$ of $H$ are real, $\lambda \in \mathbb{R}, 0 \geq \lambda < 1$ for a bounded domain.

Keywords: Boundary elements, Hybrid boundary elements, Spectral properties, Variational methods

1. INTRODUCTION

A variationally based counterpart of the boundary element method was introduced in 1987 (Dumont, 1987, 1989, 2003). Starting point was the generalized expression of the total potential energy, to which displacement compatibility assumptions had to be explicitly added, so clarifying the discussion on the symmetry characteristics of the resultant equations: symmetry or nonsymmetry is a matter of adequate or inadequate variational treatment of the boundary conditions. The variationally consistent, generalized displacement formulation thus obtained turned out to be equivalent to the formulation based on the Hellinger-Reissner potential (Reissner, 1950), exactly as Pian had developed for finite elements (Pian, 1964). In allusion to Pian’s work this new method was baptized the hybrid boundary element method – HBEM. In fact, the name "hybrid finite element method" had been coined by Pian in 1967 "... to signify elements which maintain either equilibrium or compatibility in the element and then to signify elements which maintain the interelement boundary" (Pian, 1983). According to Oden and Reddy (1976), Pian has formulated a hybrid displacement method, since it results in a stiffness matrix.

The following developments apply to elasticity problems, in general. The HBEM relies on a kinematic transformation matrix that turns out to be the same double layer potential matrix $H$ of the conventional, collocation boundary element method. For a bounded domain $\Omega$, the null space $N(H)$ spans the subspace of rigid body displacements, which are always easily representable. Moreover, whenever required, equations based on $N(H)$ always lead to well-conditioned systems.

In the HBEM, however, the properties of $N(H^T)$ are the ones specifically required, and they may turn out difficult to understand, even for apparently simple numerical examples, possibly leading to extremely ill-conditioned systems, although always expressing sound mechanical properties. Such properties have demanded several years of investigation before they could be fully understood. The unfolded mathematical properties of $H^T$ – and how they fit in a variational framework – are in fact elegant and beautiful. For the sake of simplicity, the most important features of $H^T$ are addressed starting from a simple virtual work statement that, although implicit in the Hellinger-Reissner potential, may be established on a completely independent context. The properties that come out from the present developments may become useful in the evaluation of results also in the frame of the CBEM (Dumont, 1989; Gaul et al., 1998). They are presented herein as linear algebra features per se, independently from applications that may be derived therefrom (Dumont, 2010).

In the following, the matrix $H$ is introduced in the frame of a two-field virtual work principle.

2. DISPLACEMENT VIRTUAL WORK FOR EQUILIBRIUM CHECKING

One performs the static analysis of an elastic body submitted to traction forces $\bar{t}_i$ on part $\Gamma_\sigma$ of the boundary and to displacements $\bar{u}_i$ on the complementary part $\Gamma_u$. For the sake of simplicity, body forces are not included. The present formulation applies in principle to a finite, simply-connected open domain $\Omega$ with piece-wise smooth boundary $\Gamma$ that does not have cusps (Chen and Zhou, 1992). Unbounded and multiply-connected regions are dealt with later on in terms of the domain $\bar{\Omega}$ defined as the complement of the closure of $\Omega$. Thus, holes are ultimately comprised by the formulation. Internal and edge cracks may also be included in the numerical model provided that ad-hoc functions take the local stress gradients into account (Dumont and Lopes, 2003), a subject that is beyond the present scope, although the implications of such topological issues are addressed, as they are uncovered as particular properties of $N(H^T)$. 
For notation conciseness, the arguments \( x, y, z \) of the functions are omitted. Indicial notation is used, with subscripts \( i, j \) assuming values 1, 2 or 3, as referred to the global coordinates. A subscript after a comma denotes derivative with respect to the corresponding coordinate direction. Repeated subscripts indicate summation. Arrays of constants are also referred to without subscripts, using bold-face, capital letters for matrices, and bold-face, lower letters for vectors.

### 2.1 Proposition on stress and displacement representations

The numerical model is formulated in terms of two fields, as subsequently described in detail:

- One field for displacements on \( \Gamma \), given as functions of \( n^d \) nodal displacement parameters \( \mathbf{d} = [d_n] \in \mathbb{R}^{n^d} \) located along \( \Gamma \). These functions have local support and satisfy the displacement boundary conditions as a premise. This parametric representation is referred to as the external, or displacement, reference system.

- A second field for stresses in \( \Omega \), given as a series of functions of \( n^* \) point force parameters \( \mathbf{p}^* = [p^*_n] \in \mathbb{R}^{n^*} \) located along \( \Gamma \). These functions have global support and satisfy the domain equilibrium equations as a premise. This parametric representation is referred to as the internal, or force, reference system.

A displacement virtual work principle will provide the conceptual means of correlating these two fields, in a way that naturally leads to the matrix \( \mathbf{H} \). Most important, the formulation makes use of \( n^* = n^d \) singular fundamental solutions for the stress representation in \( \Omega \). The following strict proposition is of paramount importance to arrive at the matrix \( \mathbf{H} \), as developed both in the CBEM and in the HBEMs, as well as for its adequate mathematical assessment.

**Proposition 1** The parameters \( \mathbf{d} \) and \( \mathbf{p}^* \) are identified with nodal displacements and forces that act at the same points distributed along the boundary \( \Gamma \) and are equally oriented, in such a way that \( \mathbf{p}^* \mathbf{d} \) has the meaning of (virtual) mechanical work. There are \( n^\text{ndof} \) displacement and force degrees of freedom at each nodal point.

\( n^\text{ndof} = 1, 2 \) or 3, for general problems of potential, 2D elasticity and 3D elasticity, respectively. One might in principle develop a formulation with any number of parameters attached to each nodal point (Tomlinson et al. 1995). Another possibility would be to have \( \mathbf{d} \) and \( \mathbf{p}^* \) with no clear mechanical meaning at all, just as discretization parameters (Dumont and Prazeres, 2005). Specific problems, such as in the bending theory and in the gradient elasticity, may demand higher \( n^\text{ndof} \) for a meaningful numerical discretization.

#### 2.2 Displacement assumption on \( \Gamma \)

The displacements \( u_i \) are approximated along \( \Gamma \) by \( u_i^d \) given as

\[
  u_i^d = u_{in} d_n \quad \text{on} \quad \Gamma
\]

(1)

where \( u_{in} \) are polynomial interpolation functions with compact support and \( d_n \) are nodal displacement parameters. The boundary geometry is approximated from the nodal attributes using the same interpolation functions \( u_{in} \) (isoparametric representation). Evaluated at nodal points, the interpolation functions \( u_{in} \) present the property

\[
  u_{in} \equiv \delta_{in}
\]

(2)

for \( i \) referring to a degree of freedom and the generalized Kronecker delta defined as:

\[
  \delta_{in} = \begin{cases} 
  1 & \text{if } i \text{ and } n \text{ refer to the same degree of freedom} \\
  0 & \text{otherwise}
\end{cases}
\]

(3)

Equation (1) is supposed to hold along \( \Gamma_u \), in particular: \( \pi_i = u_{in} \bar{d}_n \), in which \( \bar{d}_n \) are nodal values of \( \pi_i \).

#### 2.3 Stress assumption in \( \Omega \)

In the absence of body forces, the stress field in \( \Omega \) is approximated by a sum of homogeneous solutions,

\[
  \sigma_{ij} = \sigma_{ijm}^* p_m^*
\]

(4)

where \( p_m^* \) are force parameters and \( \sigma_{ijm}^* \) are singular fundamental solutions with global support and analytical in \( \Omega \) (\( \sigma_{ijm}^* \) may be called, in the present context, interpolation, approximation or trial functions). Although omitted, the arguments of the fundamental solutions are \( (x - x_M, y - y_M, z - z_M) \), where \( (x_M, y_M, z_M) \) is the source point – at which the point force \( p_m^* \) is applied – and \( (x, y, z) \) is the field point where the effect of \( p_m^* \) is evaluated. A fundamental solution fulfills following equilibrium equations as a premise:

\[
  \begin{cases} 
  \sigma_{ijm}^* = 0 \\
  \sigma_{ijm}^* = \sigma_{jim}^* 
\end{cases} \quad \text{in} \quad \Omega
\]

(5)
Except for analyticity, no concern is explicitly made about equilibrium of the fundamental solutions on \( \Gamma_\sigma \), although some boundary conditions might be included as a premise, too (Green’s functions). According to Proposition 1, the fundamental solutions \( \sigma^*_{jim} \) turn out to be singular just outside \( \Omega \), more precisely at points of application of \( p^*_m \) distributed along \( \Gamma \), so that one might express instead of the first line of Eq. (5):

\[
\sigma^*_{jim,j} + \Delta_{im} = 0 \quad \text{in} \quad \Omega_{\text{ext}} = \Omega \cup \Omega_0
\]

This is the usual expression found in the literature on boundary integral equations, except that, for clarification of concepts, one also uses in this paper the extended domain \( \Omega_{\text{ext}} = \Omega \cup \Omega_0 \), where \( \Omega_0 \) is understood as a set of infinitesimally small, closed regions containing each point of singularity. \( \Delta_{im} \) is a pulse function, which has zero value everywhere in the domain, except for the vicinity of the point of application of \( p^*_m \), where it tends to infinity. According to Proposition 1, one normalizes \( \Delta_{im} \), for a domain \( \Omega_{\text{ext}} \) comprising a singularity:

\[
\int_{\Omega_{\text{ext}}} \Delta_{im} \, d\Omega \equiv \int_{\Omega_0} \Delta_{im} \, d\Omega = \delta_{im}
\]

for \( \delta_{im} \) defined as in Equation (3). Thus, \( p^*_m \) in Equation (4) has the meaning of a point force applied at a nodal point on \( \Gamma \), according to Proposition 1, with \( m \) characterizing both a geometric location and a direction.

\( \Omega \) is the actual domain of interest and there is in principle no need to invoke the extension \( \Omega_{\text{ext}} \). The singularity explicitly expressed by \( \Delta_{im} \) in Equation (6) is a welcome feature, as it assures that the resulting equation systems are well conditioned. When no singularity occurs, there is no need to make a distinction between \( \Omega \) and \( \Omega_{\text{ext}} \), or either between \( \Gamma \) and \( \Gamma_{\text{ext}} \), which enclose \( \Omega \) and \( \Omega_{\text{ext}} \), respectively.

### 2.4 Displacement virtual work

In the absence of body forces, equilibrium of the stress field \( \sigma^*_j \) with forces \( \overline{T}_i \) along \( \Gamma_\sigma \) occurs if and only if

\[
\int_{\Omega} \sigma^*_j \delta u^d_j \, d\Omega = \int_{\Gamma_\sigma} \overline{T}_i \delta u^d_i \, d\Gamma
\]

for an arbitrary, virtual field of displacements \( \delta u^d_j \) such that \( \delta u^d_j = \frac{1}{2} (\delta u^d_j + \delta u^d_j^*) \) in \( \Omega \) and \( \delta u^d_j = 0 \) on \( \Gamma_u \). The stress field \( \sigma^*_j \) approximated according to Equation (4) already satisfies Equation (5). Taking into account that \( \delta u^d_0 = 0 \) on \( \Gamma_u \), the boundary integral of Equation (8) may be extended to the whole boundary \( \Gamma \). Integration by parts of the term on the left-hand side of Equation (8), application of Green’s theorem and use of Eqs. (1) and (4) lead to the matrix expression

\[
H^T \mathbf{p}^* = \mathbf{p}
\]

where \( H = [H_{mn}] \in \mathbb{R}^{n^* \times n^d} \) and \( \mathbf{p} = [p_n] \in \mathbb{R}^{n^d} \) are expressed, for a bounded domain, as

\[
H_{mn} = \int_{\Omega} \sigma^*_{jim} n_j u_{in} \, d\Gamma \equiv \int_{\Gamma_{\text{ext}}} \sigma^*_{jim} n_j u_{in} \, d\Gamma + \delta_{mn} \equiv \int_{\Gamma_p} \sigma^*_{jim} n_j u_{in} \, d\Gamma + \int_{\Gamma_{\text{disc}}} \sigma^*_{jim} n_j u_{in} \, d\Gamma
\]

\[
p_n = \int_{\Gamma} t_i u_{in} \, d\Gamma
\]

\( \mathbf{p} \) is the vector of equivalent nodal forces, which is in part known (on \( \Gamma_\sigma \) and must be in part evaluated (as reaction forces along \( \Gamma_u \)), for a general mixed boundary problem. \( H \) is the same double-layer potential matrix obtained in the collocation boundary element method (Brebbia et. al, 1984). \( \delta_{mn} \) is the identity matrix \( I \) of order \( n^* = n^d \). The evaluation of \( H \) is straightforward also for unbounded and multiply-connected domains, with the boundary unit normal \( \overline{n} \) always pointing outward. The boundary integral of Equation (10) is singular, for \( n \) and \( m \) referring to the same nodal point, but may be completely evaluated either mathematically – split into a Cauchy principal value (in terms of a finite-part integral) and a discontinuous term – or by using rigid body displacements, according to Eq. (15). Figure 1 illustrates the three mathematical equivalences stated in Eq. (10) for the boundary segment in the vicinity of a singularity point, in which it is seen that the singularity is outside the domain \( \Omega \). Application of the virtual work principle of Eq. (8) to an unbounded domain is straightforward. Figure 2 illustrates Eq. (10) applied to an infinitely large domain \( \Omega \) – complementary to the closure of \( \Omega \) – enclosed externally by a single boundary \( \Gamma_{\infty} \) and internally by the same boundary of Fig. 1, but characterized as \( \Gamma \), since the outward normal \( \overline{n} \) is reversed. As illustrated, the finite-part integrals for \( \Gamma \) and \( \overline{\Gamma} \) have the same magnitude and reversed signals. The discontinuous terms for \( \Gamma \) and \( \overline{\Gamma} \) add to \( \delta_{im} \), as a result of Eq. (7). Then, if one denotes the result of Eq. (10) for the complementary domain \( \overline{\Omega} \) as \( \overline{H} \), a corresponding expression may be written as

\[
\overline{H}^T \mathbf{p}^* = \overline{\mathbf{p}}\quad \text{where} \quad \overline{H}^T = I - H^T
\]

The coefficients of \( \overline{\mathbf{p}} \) for prescribed tractions are evaluated along \( \Gamma_\sigma \).
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3. LINEAR ALGEBRA PROPERTIES OF THE TRANSFORMATION MATRIX $H$

As formulated, $H^T$ is an equilibrium matrix that transforms nodal forces $p^*$ of the internal reference system into equivalent nodal forces $p$ of the external reference system. One also obtains in the complete frame of the Hellinger-Reissner potential that $H$ is a kinematic matrix that transforms nodal displacements $d^*$ of the external reference system into equivalent nodal displacements $d$ of the internal reference system. This is expressed as follows, already extending the concepts to $\Omega$:

$$d^* = Hd \iff p = H^T p^*; \quad d = H^T d \iff \overline{p} = H^T \overline{p}^*$$

(13)

One obtains from Eqs. (12) and (13) that

$$d^* + \overline{d} = \overline{d} \quad \text{and} \quad p + \overline{p} = p^*$$

(14)

Let the columns of the matrix $W \equiv N(H) = [W_{ns}] \in \mathbb{R}^{n \times n^*}$ be a basis of the set of $n^*\times n^*$ rigid body displacements that the bounded body may suffer, as measured at the nodal points. $W$ depends only on the body geometry. One may also define a matrix $\mathbf{V} \equiv N(H^T) = [V_{ns}] \in \mathbb{R}^{n^* \times n^*}$. Differently from $W$, $\mathbf{V}$ depends on $u_{in}$ and $\sigma^*_{in}$ (and thus from the material properties of the elastic problem). One may write:

$$H W = 0; \quad H^T V = 0$$

(15)

$$\overline{H} W = W; \quad \overline{H}^T V = V$$

(16)

Owing to space restrictions, following theorems are given without proof. They are detailed in the full version of the paper.

**Theorem 1 (Evaluation of $V$)** If $H$ is nondefective, then both $W^T V$ and $H + WW^T$ are nonsingular. Then, a simple mean of evaluating $V$, whenever required, is by solving the equation system $(H + WW^T) V = W$.

**Theorem 2 (Contraction)** $H$ is a contraction (Ben-Israel and Greville, 1980), and so is $H^T$, since from Eqs. (13) and (14), for any norm $\| \cdot \|$, $\| p \| = \| H^T p^* \| \leq \| p^* \|$. $\| \overline{p} \| = \| \overline{H}^T \overline{p}^* \| \leq \| \overline{p}^* \|$.

**Corollary 1** $H$ singular $\Rightarrow \overline{H}$ nonsingular.

**Theorem 3 (Diagonal of $H$)** $0 < H_{ij} < 1, \quad 0 < \overline{H}_{ij} < 1 \quad \text{for all} \ j = 1, \ldots, n^* = n^d$.

**Theorem 4 (Positive direction)** $p^T p^* > p^T p \geq 0, \quad p^T p^* > p^T \overline{p} > 0$.

**Theorem 5 (Real eigenvalues)** All eigenvalues $\lambda$ and eigenvectors $\phi$ of $H$ are real. Moreover, for any $\lambda$: $0 \leq \lambda < 1$.

Theorem 4 follows from Proposition 1 and the displacement and stress assumptions, or simply from Theorem 3. A proof of Theorem 5 follows from Theorem 4, which must be valid for any eigenvector $\phi$ of $H^T$. Suppose that $\phi$ is complex, with corresponding complex eigenvalue $\lambda$, and let $(\cdot)^*$ only in the following outline mean the Hermitian adjoint. One obtains from $\phi^* H^T \phi \geq 0$ and Theorem 2 that $0 \leq \Re(\lambda) < 1$. The proof that $\Im(\lambda) = 0$ deserves more attention. A strict proof takes into account some degenerate cases as when symmetry assumptions are included in the formulation.
Definition 1 (Oblique and orthogonal projectors) Let $W, V \in \mathbb{R}^{n_d}$ be subspaces spanned by the columns of $W, V$, and let the subspaces $W^\perp, V^\perp \in \mathbb{R}^{n^d}$ spanned by the columns of $W^\perp, V^\perp \in \mathbb{R}^{n_d \times (n^d - n^d)}$, be orthogonal to $W$ and $V$, respectively. Also, let $V^T W$ be nonsingular. $P_{W \perp V^\perp} = W(W^T W)^{-1} W^T$ is an idempotent matrix, called the projector on $W$ along $V^\perp$, or oblique projector $P_W \equiv P_{W \| V^\perp} = W(W^T W)^{-1} W^T$ is the orthogonal projector on $W$.

Theorem 6 The following are equivalent expressions of the $\{1, 2, 3, 4\}$, or Moore-Penrose inverse of $H$:

\[
H^1 = W^\perp \left(V^T H W^\perp\right)^{-1} V^T
\]

\[
H^1 = H^T (H^T H + P_V)^{-1} P_{V^\perp}; \quad H^1 = P_{W \perp} (H^T H + P_W)^{-1} H^T
\]

\[
H^1 = P_{W^\perp} (H + P_{V^\perp})^{-1} P_{V^\perp W}; \quad H^1 = (H + P_{V^\perp})^{-1} P_{V^\perp}
\]

\[
H^1 = P_{W^\perp} (H + P_V)^{-1} P_{V^\perp}; \quad H^1 = P_{W \| V^\perp}
\]

The expressions on the fifth row only apply if $H$ is nondefective. See Ben-Israel and Greville (1980) for the theory of generalized inverses.

In case of Neumann boundary conditions, the complete solution of a problem (except for rigid body displacements) is obtained by solving Eq. (12) for an unbounded domain, as $H^T \tilde{H}$ is nonsingular, or Eq. (9) for a bounded domain, subjected to the restriction that $V^T p^* = 0$, which is the same as applying the transpose of any of the inverses given in Theorem 6.

Given a submatrix $W_N = [W_{N \| \text{ndof}}] \in \mathbb{R}^{n_{\text{ndof}} \times n_{\text{ndof}}}$ of $W$ related to a nodal point $N$, then $W_N^T W_N$ is always nonsingular and well conditioned, independently from problem topology and from material properties. Then, $H W = 0$, Eq. (15), for a bounded domain is a means to evaluate the coefficients of $H$ about its main diagonal, thus circumventing the need to deal with the singular boundary integral of Eq. (10). A similar property is attached to $V$, although involving far reaching concepts.

Theorem 7 If a numerical model converges to the idealized mechanical problem with increasing mesh refinement, then it is possible to have a boundary mesh such that, for all points in $\Omega$ not too close to the boundary nodes, $\|\sigma_{ijm} V_{mr}\| \leq \epsilon$ for an arbitrarily small error $\epsilon \geq 0$.

This theorem (Dumont, 2010) follows from the fact that $V$ is a basis of point forces $p^*$ that are not in equilibrium and therefore cannot generate a stress state (Dumont, 1989). This theorem may be used in a reverse way, according to the following Proposition.

Proposition 2 For points in $\Omega$ that are too close to a boundary node, so that the boundary layer effect becomes perceptible (or at the boundary node itself, which is actually outside $\Omega$, as illustrated in Figs. 1 and 2), the term $\sigma^*_{ijm}$ of the fundamental solution in Eq. (4) that tends to increase without bounds may be consistently replaced with finite terms evaluated in such a way that $\|\sigma^*_{ijm} V_{mr}\| = \text{minimum}$ for elasticity problems in general.

Let $V_N = [V_{N \| \text{ndof}}] \in \mathbb{R}^{n_{\text{ndof}} \times n_{\text{ndof}}}$ be a submatrix of $V$ related to a nodal point $N$. The use of Proposition 2 to obtain results at a point close to or at nodal point $N$ in terms of least squares is only possible if $V_N^T V_N$ is well conditioned. However, this is not the case for points close to or at crack or notch tips, strong concavities and cavities, as well as for symmetry axes or planes in case of numerical problems that have symmetries embedded in the formulation, not to mention some unpredictable locations on a neat convex domain that contains a non-homogeneous material (Dumont, 2008). In fact, a problem in a convex domain with non-homogeneous properties may be expressed via a Kirchhoff-like transform as an equivalent one with homogeneous properties: the transformed geometry may present strong concavities. It is amazing to find out that the problem formulated from the proposition that $\|\sigma^*_{ijm} V_{mr}\|$ is a minimum is always consistent, but involves, in case of strong local non-convexities, a division between small numbers that are in the range of the discretization error, which still validates Theorem 7 but then becomes of no help (Dumont, 2008).

The explanation of this fact is extremely simple, although its has taken many years of investigation to become clear. For points in a region far from the boundary or close to a strictly convex part of the boundary, the stress gradient may be globally described and Proposition 2 is, whenever necessary, a means of eliminating the spurious gradient locally introduced by the singular fundamental solution. On the other hand, a region close to crack or notch tips, for instance, has a stress gradient that is locally conditioned and must be described adequately. In such a case the ill-conditioning of $V_N^T V_N$ only tells that the mathematics of the problem is not contradicting the local mechanical peculiarities.

This problem has astonished the author and his collaborators from the very beginning of the development of the HEM. However, it has been completely understood and explored in its mathematical elegance and appropriateness. Depending on the problem, the local stress gradient must be dealt with directly. There are in general sound mechanical alternatives to circumvent ill-conditioning problems related to $V_N^T V_N$ (Oliveira and Dumont, 2009; Dumont, 2010).

As shown in the extended version of this paper, the spectrum $\sigma(H)$ characterizes if a domain is bounded or unbounded, simply or multiply connected. Together with $\|V_N^T V_N\|$, for $N$ spanning all the boundary nodal points of the numerical model, one precisely identifies cavities and concavities, in general, as well as embedded symmetries.
4. CONCLUSIONS

The second Section of this paper is on purpose relatively extensive, as the conceptual basis of the double layer potential matrix $H$ must be well established before one starts the outline of its linear algebra properties. In fact, one may conceive a boundary element formulation – variational or not – in which the boundary interpolation functions $u_{ij}$ or the singular fundamental solutions $\sigma_{ijm}^*$ are not according to Eqs. (2) and (7), and such that Proposition 1 is not observed. A feasible boundary element formulation (in which $H$ is defective, for instance) might still be developed, although lacking generality and several properties that render the equations well conditioned. The “regularized boundary element methods”, in which the source point is located at a finite distance outside the domain, are just an example of formulation for which the matrix $H$ has no clear and sound properties. The mathematical properties, for the problem as stated, owe their elegance to two unrelated singularities: of the fundamental solutions and in terms of linear algebra. Not coincidentally, the single layer potential matrix $G$ of the boundary element methods also features interesting linear algebra properties that still deserve a deeper investigation (Dumont, 1998). The linear algebra properties of the matrix $H$, as briefly presented – and particularly the ones related to the problem’s topology –, shall be outlined in detail in the extended version of this manuscript.

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6. REFERENCES


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