APPLICATION OF THE BOUNDARY ELEMENT METHOD TO VISCOELASTIC AND VISCOPLASTIC ANALYSIS

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Abstract. In the current work two time-dependent boundary element method formulations are shown. The first one is developed to an analysis on a viscoelastic media. The Kelvin-Voigt viscoelastic material model is chosen to simulate the time-dependent behavior. The second one deals with an analysis on a viscoplastic media. An elasto-viscoplastic material model is chosen alongside Perzyna’s flow rule to simulate the time-dependent behavior. A procedure to automatic generate cells in the zones of the domain that yielded is used. Examples using both formulations are analyzed and the results are compared with the exact and Finite Element Method (FEM) results.

Keywords: boundary element method, viscoelasticity, viscoplasticity, automatic cell generation

1. INTRODUCTION

Many materials, such as concrete, polymers and soils, exhibit a time-dependent behavior, i.e. for a constant load different deformation and stress patterns are developed within the body as time passes. In many projects this kind of material behavior has to be taken into account. In the current work, two well known mechanical models are used to simulate this time-dependent behavior. When treating the viscoelastic responses, the Kelvin-Voigt model is used, while an elasto-viscoplastic model together with Perzyna’s flow rule is applied for viscoplastic responses.

The Boundary Element Method (BEM) has been chosen as the numerical tool to perform the analysis. When the possibility of material flow is considered in a BEM analysis (as in elastoplasticity and viscoplasticity) the domain zones where the flow occurs also need discretization. Usually the discretization of these zones has to be done a priori, what means more data to be given by the user. This drawback is avoided by using a procedure to automatically generate the discretization of the yielded zones during the analysis.

2. CONSTITUTIVE EQUATIONS

The uniaxial representations of the Kelvin-Voigt and the elasto-viscoplastic mechanical models are shown in Fig. 1. Their uniaxial constitutive equations can be written respectively as (Fung, 1965 and Venturini, 1983)

\[ \sigma = E \varepsilon + \mu \dot{\varepsilon} \]  
\[ \Delta \sigma = E \Delta \varepsilon - E \Delta \varepsilon^{vp} \]

where \( \sigma, \varepsilon, \varepsilon^{vp}, \dot{\varepsilon}, E \) and \( \mu \) stands for the stress, the strain, the viscoplastic strain, the strain time derivative, an elastic parameter and a viscous parameter, respectively. Equation (2) is defined in an initial strain sense and is valid only for a small increment of time \( \Delta t \).

![Uniaxial Constitutive Models](image)

Figure 1 - Uniaxial Constitutive Models

In three dimensions, for an isotropic material, due to the similarities between the fourth order constitutive tensors, the viscoelastic tensor \( \mu_{ijlm} \) can be written as a function of the elastic tensor \( C_{ijlm} \) using a constant \( \gamma \) as
\[ \mu_{ijlm} = \gamma C_{ijlm} \]  

Then, for tridimensional problems Eq. (1) and Eq. (2) can be written, respectively, as

\[ \sigma_{ij} = C_{ijlm} (\gamma \dot{e}_{lm} + e_{lm}) \]  

\[ \Delta \sigma_{ij} = C_{ijlm} (\Delta e_{lm} - \Delta e_{lm}^{vp}) \]  

where \( \sigma_{ij}, e_{lm}, e_{lm}^{vp} \) are, respectively, the stress, the strain, the viscoplastic strain, the strain time derivative tensors. Equation (5) also is valid only for a small increment of time \( \Delta t \).

3. VISCOPLASTIC YIELD FUNCTION AND FLOW RULE

In a similar way as the elastoplastic analysis, a yield function and a flow rule are needed. The von Mises yield criterion and Perzyna’s viscoplastic flow rule are adopted in order to perform the viscoplastic analysis and can be written, respectively, as

\[ F = \sqrt{3J_2} - Y(h) \]  

\[ \Delta e_{lm}^{vp} = \frac{1}{\eta (\frac{\alpha}{F_0})} \frac{\partial F}{\partial \sigma_{lm}} \Delta t \]  

where \( J_2 \) is the second invariant of the stress deviatoric tensor, \( Y \) is a function of the hardening parameter \( h \), \( \eta \) is a viscous parameter, \( F_0 \) denotes any convenient reference value of \( F \) for the dimensionless representation of \( \Delta e_{lm}^{vp} \). The \( (\cdot) \) notation in Eq. (7) implies that if \( F < 0 \), \( (F/F_0) = 0 \).

Then, for an increment of time \( \Delta t \), an increment of the residual/initial stress tensor \( \sigma_{ij}^{\alpha} \), can be defined as

\[ \Delta \sigma_{ij}^{\alpha} = C_{ijlm} \Delta e_{lm}^{vp} \]  

Unlike in the viscoelastic analysis, the time increment value cannot be chosen freely in the viscoplastic analysis. In the current work, the time step length proposed by Cormeau (1975) for the associative von Mises flow rule is used and can be written as

\[ \Delta t \leq \frac{4\eta (1 + \nu) F_0}{3E} \]  

where \( \nu \) is the Poisson coefficient and \( E \) is the Young’s modulus.

4. BOUNDARY INTEGRAL EQUATIONS

The numerical implementation for the viscoelastic case is based on the following displacement Boundary Integral Equation (BIE) which is obtained by introducing the viscoelastic constitutive model, Eq. (4), in the formulation (Mesquita, 2002 and Carbone, 2007).

\[ C_{kl}(P) u_i(P) + C_{ki}(P) \gamma \dot{u}_i(P) = \int_{\Gamma} U_{kl}^{\alpha}(P, Q) p_i(Q) d\Gamma - \int_{\Gamma} T_{kl}^{\alpha}(P, Q) u_i(Q) d\Gamma - \gamma \int_{\Gamma} T_{kl}^{\alpha}(P, Q) \dot{u}_i(Q) d\Gamma \]  

where \( U_{kl}^{\alpha} \) and \( T_{kl}^{\alpha} \) are, respectively, the displacement and traction Kelvin fundamental solutions, and \( u_i, \dot{u}_i, p_i \) are respectively the displacement, displacement time derivative and traction components.

On the other hand the viscoplastic analysis has been developed from the following incremental displacement BIE (Venturini, 1983)

\[ C_{kl}(P) \Delta u_i(P) = \int_{\Gamma} U_{kl}^{\alpha}(P, Q) \Delta p_i(Q) d\Gamma - \int_{\Gamma} T_{kl}^{\alpha}(P, Q) \Delta u_i(Q) d\Gamma + \int_{\Omega} E_{kij}^{\alpha}(P, q) \Delta \sigma_{ij}^{\alpha}(q) d\Omega \]  

where \( E_{kij}^{\alpha} \) is the strain Kelvin fundamental solution. The viscoplastic relationship, Eq. (5), has been taken into account in this BIE. This is the same BIE of the elastoplastic formulation based on initial stress; the main difference is that an
elastoplastic analysis proceeds in load increments while a viscoplastic analysis proceeds in time increments (Beer et al. 2008).

5. BOUNDARY ELEMENT METHOD

By considering the difficulties involved in solving Eq. (10) and Eq. (11) with closed form solutions, the boundary is then discretized into boundary elements defined by nodes, following the usual procedures of the BEM. The boundary values (displacement and tractions) are approximated using polynomial functions based on the respective boundary nodal values. Bearing in mind the viscoplastic analysis, the domain is discretized into internal cells defined by internal nodes. In the current work continuous linear and quadratic boundary elements and cells are used for the approximations.

5.1 Viscoelastic Formulation

By applying the discretization and approximations in the viscoelastic BIE, Eq. 10, one can write a temporal system of algebraic equations as follows

\[ [T][u] + \gamma [T][\dot{u}] = [U][p] \]  

(12)

where \{u\}, \{\dot{u}\} and \{p\} are respectively vectors containing the displacements, displacement derivatives and tractions boundary nodal values.

By considering a linear approximation of the displacement time derivative, i.e.

\[ \dot{u}_{s+1} = \frac{u_{s+1} - u_s}{\Delta t} \]  

(13)

The following system of equation can be written

\[ (1 + \gamma \frac{\dot{u}}{\Delta t}) [T][u_{s+1}] = [U][p_{s+1}] + \{F_s\} \]  

(14)

This system of equations has to be solved for each time step where \{F_s\} refers to the values of the last time step, which can be evaluated using

\[ \{F_s\} = \gamma [T] \frac{u_s}{\Delta t} \]  

(15)

5.2 Viscoplastic Formulation

Applying the discretization and the respective approximations in the viscoplastic BIE, Eq. (11), leads to an incremental system of equations as follows

\[ [T]\Delta u = [U]\Delta p + [E] \Delta \sigma^o \]  

(16)

where \{\Delta \sigma^o\} is the vector containing nodal values of the residual stress increment.

5.3 Singular Integration and Stress Evaluation

At the boundary, the weakly singular integrals are evaluated by means of the logarithmic Gaussian quadrature, while the strongly singular integrals are treated using a numerical technique proposed by Guiggiani and Casalini (1987) to directly calculate the Cauchy principal value. Regarding domain integration, weakly singular integrals are evaluated using cell subdivision (Gao and Davies, 2002) and strongly singular integrals are computed using subtraction of singularity and semi analytical integration as presented by Gao and Davies (2002).

The stresses at interior points are evaluated using Somigliana stress identity with initial stress. While at the boundary, the stresses are evaluated using a methodology called ‘stress recovery method’ (Gao and Davies, 2002) avoiding the evaluations of hyper singular integrals.

5. DOMAIN AUTOMATIC DISCRETIZATION

The BEM implementation in this work requires a priori only the boundary discretization of the problem. The domain discretization in cells during the viscoplastic analysis is performed only in the yielded regions by applying the algorithm proposed by Ribeiro et al. (2008). For each time step the yield criteria is checked at all nodes and then cells
are created around each yielded node until all these nodes are surrounded by nodes yet in an elastic stress state. This scheme is illustrated in Fig. 2 from (a) to (d).

![Update and Evolution of the Discretized Region](image)

**Figure 2 - Update and Evolution of the Discretized Region.**

### 6. COMPUTATIONAL ROUTINES

#### 6.1 Viscoelastic Routine

The procedure to solve Eq. (14) consists mainly of the following steps:

(i) The matrices $[T]$ and $[U]$ are evaluated, as in the classical BEM.

(ii) Using the time step $\Delta t$ adopted and the boundary conditions, Eq. (14) is rearranged as

$$\{x_{s+1}\} = [A]^{-1} \cdot \{(B) + \{F_i\}\}$$

where matrix $[A]$ contains the coefficients correspondent to the unknown boundary values while vector $\{B\}$ gives the influence of the prescribed boundary values.

(iii) Then a loop over time increments ($\Delta t$) starts, where in each time increment:

- The $\{F_i\}$ vector is evaluated using Eq. (15).
- Then $\{x_{s+1}\}$ is evaluated by Eq. (17).
- All the unknowns are updated.

This loop continues until the total time interval is reached.

#### 6.2 Viscoplastic Routine

The procedure to solve Eq. (16) consists mainly of the following steps:

(i) The matrices $[T]$ and $[U]$ are evaluated, as in the classical BEM.

(ii) Using the boundary conditions, Eq. (16) is rearranged as

$$\{\Delta x\} = [A]^{-1} \cdot \{(B_0 + \Delta f)\}$$

where $\{\Delta f\} = [E] \cdot [\Delta \sigma^0]$.

(iii) The system in Eq. (18) is solved for $\{\Delta x\}$, disregarding $\{\Delta f\}$, and $\{x\} = \{\Delta x\}$.

(iv) Then a loop over time increments $\{(\Delta t)\}$ starts, where in each time increment:

- The stress is evaluated in each node and the yield criterion is checked.
- Cells are created around all yielded nodes and the correspondent $[E]$ matrix is evaluated.
- A $\{\Delta \sigma^0\}$ vector is evaluated using Eq. (8).
- Then an $\{\Delta x\}$ vector is evaluated by Eq. (18), now disregarding $\{B\}$.
- All the unknowns are updated, i.e $\{x\} = \{x\} + \{\Delta x\}$

This loop continues until the total time interval is reached.

### 7. VISCOELASTIC EXAMPLE

The viscoelastic problem here considered is a cylindrical tube under uniform internal pressure. Due to the double symmetry of this problem only a quarter of the tube section is analyzed. Plain strain condition is assumed. The geometry of the problem, the parameters and the boundary element mesh used are shown in Fig. 3. For the Kelvin-Voigt viscoelastic solid model as time tends to the infinite, the viscoelastic solution tends to the elastic solution. Thus to compare the efficiency of the implemented code we use the exact elastic solution (Mal and Singh, 1991) together with the exact viscoelastic response of the Kelvin-Voigt model (Fung, 1965).
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Figure 3 - Geometry, parameters and the boundary element mesh.

The final results (i.e. when $t \to \infty$) can be compared in Table 1. Two boundary element meshes where used. The exact time dependent behavior and the time dependent behavior for arbitrary time steps values can be compared in Fig. 4. Since the coarser mesh already presents a very good result, in Fig. 4 only the results of this mesh are shown. If one is interested only on the final result the time step value can be chosen freely. However, as can be seen by Fig. 4, the viscoelastic behavior during the analysis is greatly influenced by the time step chosen. Several time steps have been tested to check the accuracy of the numerical BEM answers. As the time step was reduced the numerical answer became closer to the analytic solution.

Table 1 - Comparison of maximum displacement as time tends to the infinite.

<table>
<thead>
<tr>
<th>Mesh</th>
<th>Maximum Displacement [m]</th>
<th>Difference [%]</th>
</tr>
</thead>
<tbody>
<tr>
<td>Exact Elastic Solution</td>
<td>-</td>
<td>0.0095333</td>
</tr>
<tr>
<td>BEM - Viscoelastic (10 quadratic elements, 20 nodes)</td>
<td>0.0095275</td>
<td>-0.061%</td>
</tr>
<tr>
<td>BEM - Viscoelastic (20 quadratic elements, 40 nodes)</td>
<td>0.0095329</td>
<td>-0.004%</td>
</tr>
</tbody>
</table>

Figure 4 - Displacement results: Exact solution and for arbitrary time steps values.

8. VISCOPLASTIC EXAMPLE

Here is considered a perforated plate under uniform tensile load as the viscoplastic example (see Fig. 5). Due to the symmetry of this problem only half of the problem is analyzed. Plain strain condition is assumed. The geometry of the problem and the parameters used are shown in Fig. 5, where $\sigma_y$ is the yielding stress.

The final results of a viscoplastic analysis (i.e. when $t \to \infty$), must tend to the results of a elastoplastic analysis and, in this case, $\eta$ is immaterial and time plays the role of a fictitious variable (Corneanu and Zienkiewicz, 1974). The maximum values of the displacements and normal stresses obtained using two BEM meshes, as well as a comparison with the results of a FEM elastoplastic analysis, are presented in Table 2.
Figure 5 - Geometry and parameters.

Table 2 - Comparison of maximum displacement and normal stress as time tends to the infinite.

<table>
<thead>
<tr>
<th>Analysis</th>
<th>Mesh</th>
<th>Maximum Displacement [mm]</th>
<th>Maximum Normal Stress [MPa]</th>
</tr>
</thead>
<tbody>
<tr>
<td>ANSYS FEM - Elastoplastic</td>
<td>4500 elements, 13831 nodes</td>
<td>0.045898</td>
<td>325.53</td>
</tr>
<tr>
<td>BEM - Viscoplastic</td>
<td>Boundary: 256 linear elements, 256 nodes</td>
<td>0.046433 (+1.17%)</td>
<td>319.51 (-1.85%)</td>
</tr>
<tr>
<td></td>
<td>Domain: 112 cells, 120 nodes</td>
<td></td>
<td></td>
</tr>
<tr>
<td>BEM - Viscoplastic</td>
<td>Boundary: 128 quadratic elements, 256 nodes</td>
<td>0.046500 (+1.31%)</td>
<td>320.45 (-1.56%)</td>
</tr>
<tr>
<td></td>
<td>Domain: 36 cells, 116 nodes</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

9. CONCLUSIONS

The numerical BEM results obtained for both viscoelastic and viscoplastic analyses have demonstrated to be valid and accurate. In the viscoelastic analysis, the results have also demonstrated that convenient time step length must be chosen if one is interested in the partial results along time, otherwise significant errors may arise during the process. The algorithm to automatically generate internal cells has shown to be a very good alternative to avoid the a priori domain discretization in case of viscoplastic analysis, saving end-user input and computational processing effort, since cells are only generated where needed.

10. ACKNOWLEDGEMENTS

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11. REFERENCES


11. RESPONSIBILITY NOTICE

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