A SMOOTHING TECHNIQUE FOR DISCONTINUOUS BOUNDARY ELEMENTS RESULTS IN 2D ELASTICITY

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Abstract. The use of discontinuous elements in the boundary element method (BEM) does not provide continuous results across the boundary mesh, i.e. variables are not single valued across element interfaces. This work proposes the implementation of a smoothing technique for these elements in two-dimensional elasticity. The technique, able to achieve continuous (smoothed) results, is implemented for linear and quadratic elements. The methodology is based on least-squares fit of the values at the physical nodes. These approximations are made for patches of two elements and therefore the values of stresses or displacements at the geometric node shared by the patch can be recovered. New solutions with the same degree of interpolation of the original ones are obtained in each element from these recovered values and, consequently, a continuous solution is obtained. The efficiency of the proposed technique is checked through the numerical solution of known static 2D elasticity cases.

Keywords: Boundary elements method, discontinuous elements, variable recovery, variable smoothing.

1. INTRODUCTION

The relaxation of continuity conditions in discretization-based methods has gained impulse among several numerical methods in the last decade. The use of discontinuous elements in boundary element methods (BEM) is somewhat old, but discontinuous Galerkin finite element methods (FEM) are prime examples of the developments in this field. These approaches greatly simplify the computational implementation of the solution methods, and may increase their efficiency, particularly in nonlinear problems or problems containing discontinuous fields.

In the BEM context, there are a number of advantages in the use of discontinuous boundary elements in spite of the characteristic interelement discontinuities. Discontinuous interpolation presents $C^1$ continuity on all physical nodes, which simplifies the computation of strongly singular integrals. It also avoids the need of double nodes in cases containing corners and discontinuities in the boundary conditions. In addition, the use of discontinuous elements has already proved its efficiency in the solution of multi-domain BEM formulations and FEM-BEM couplings (Zhang and Zhang, 2002). On the other hand, the recovery of variables at the ends of the elements by simple extrapolation or by averaging the extrapolated results of two or more elements is usually inadequate.

A similar problem occurs in stress evaluation by FEM, since the optimal values must be computed at non-nodal points, resulting in a discontinuous – element by element – stress field which must be smoothed in order to recover nodal values. This is generally done as a post-processing step, and among the several techniques developed one can mention extrapolation from the Barlow points (Barlow, 1976), global and local $L_2$ smoothing (Hinton and Campbell, 1974) and the various types of superconvergent patch recovery (SPR) procedures (Zienkiewicz and Zhu, 1992), the latter being possibly the most used.

The present work presents the application of one-dimensional SPR methods to post-process results obtained by linear and quadratic discontinuous boundary elements. Although only elasticity problems are illustrated, the basic procedure is valid for any governing equation. It is shown that continuous and better defined interelement results are obtainable, thus eliminating one of the major drawbacks in discontinuous BEM formulations.

2. USING SPR METHODS IN DISCONTINUOUS BOUNDARY ELEMENTS

This section presents a smoothing procedure very similar to the SPR methods used in FEM, but devoted to recovering the results on the geometric nodes of discontinuous boundary elements. The underlying objective is to avoid direct variable extrapolation from the physical nodes to the geometric (end) nodes through element shape functions. This not only leads to inaccurate results at element extremities (particularly when the element offset is large), but also will require further weighting of the results in order to obtain the smoothed value at the shared node.

Let the domain boundary be defined by line segments, which are divided into discontinuous boundary elements. The objective of the variable recovery technique is to find a continuous field (stress or displacements) along the boundary segments using a set of recovered nodal parameters $\mathbf{u}'$:

$$\mathbf{u}' = \mathbf{N}\mathbf{u}^*$$

(1)

where $\mathbf{N}$ are the geometric interpolation functions of the elements discretizing the current segment.
In the present implementation, a first order neighborhood was used, i.e., the two elements will form the patch containing the shared geometric node. Figure 1 illustrates the patches for linear and quadratic discontinuous elements.

Figure 1. Discontinuous boundary element patches. (a) Linear elements; (b) Quadratic elements.

It is assumed that the recovered nodal values $u_p^*$ belong to a polynomial expansion $u_p^*$, which is valid over an element patch surrounding the geometric node considered. This polynomial expansion is one degree higher than the base functions $N$ and can be written as:

$$u_p^* = Pa$$  \hspace{1cm} (2)

where $u_p^*$ is any stress or displacement component ($\sigma_1$, $\sigma_2$, $\sigma_{12}$, $u_1$, $u_2$), $P$ contains the appropriate terms of a complete polynomial of order $p$, and $a$ contains generalized parameters to be determined.

The evaluation of the unknown parameters $a$ of the expansion in Eq. (2) is accomplished by a least square fit of $u_p^*$ using the element results at the sampling points (physical nodes) along the patch considered. Therefore, one has to find $a$ which minimizes the function:

$$F(a) = \sum_{i=1}^{m} (u_i(x_i) - u_p^*(x_i))^2 = \sum_{i=1}^{m} (u_h(x_i) - P(x_i)a)^2$$  \hspace{1cm} (3)

where $x_i$ are the local coordinates of the sampling points and $m$ is the total number of sampling points in the patch. Here, $m = 4$ for linear elements (two physical nodes in each element $\times$ two elements in the patch), while $m = 6$ for quadratic elements (three physical nodes in each element $\times$ two elements in the patch). The minimization of $F(a)$ leads to an algebraic system that can be solved as

$$a = A^{-1}b$$  \hspace{1cm} (4)

where

$$A = \sum_{i=1}^{m} P^T(x_i)P(x_i) \quad \text{and} \quad b = \sum_{i=1}^{m} P^T(x_i)u_h(x_i)$$  \hspace{1cm} (5)

Once the parameters $a$ are determined the recovered value is computed at any position of the patch by inserting the appropriate coordinates in Eq (2).

Figure 2. Variable recovery on a shared node. (a) Linear elements; (b) Quadratic elements.
The recovery is made only for the shared node of each path. Thus, the value for the central node in each quadratic element is calculated by the average value obtained by the two closest patches. The values in the extreme of a segment of contour are calculated by the nearest patch. Figure 2 shows graphically the variable recovery for linear and quadratic elements. Similar ideas can be used in higher order elements.

3. NUMERICAL RESULTS

In order to investigate the performance of the smoothing procedure, this section shows some results. Numerical integration was carried out using 16 Gauss points, in order to minimize the influence of quadrature errors. Dimensions, material properties, and other physical data are given without units, but they were specified to represent a compatible system of units. The material properties used in all cases are: $E = 210\times 10^6$ and $\nu = 0.3$. Plane-stress condition is assumed throughout this section.

3.1. Square-plate with a central hole under traction

A 100×100 square plate with a central hole of radius $R = 5$ was analyzed. Due to symmetry, only one quarter of the plate was considered (Fig. 3). The traction loading along the upper side was set to $P = 1$. The offset of all boundary elements used in the mesh is 15% of the element length.

Linear and quadratic elements were used with two different meshes for each type of element. Mesh 1 used an element size of 2.5 along the straight boundaries and four elements along the quarter-circle. Mesh 2 used an element size of 1.25 and eight elements along the quarter-circle.

Figure 3. Squared plate with a central hole under uniform traction.
Figure 4. Hoop stress smoothing: a) linear elements – mesh 1, b) quadratic elements – mesh 1.

Figure 5. Radial stress smoothing. Linear elements: (a) mesh 1, (b) mesh 2.

Figure 6. Radial stress smoothing. Quadratic elements: (a) mesh 1, (b) mesh 2.
3.2. L-shaped domain

Figure 7 shows the geometry and boundary conditions of an L-shaped plate. This problem is a typical test for adaptive meshing procedures in both the FEM and the BEM (Gago et al., 1983, Zhao & Wang, 1999, Zienkiewicz & Zhu, 1992b). It was also used as a numerical example to demonstrate the accuracy of a formulation for interelement stress evaluation by boundary elements (Zhao, 1996).

Due to a re-entrant corner the normal stress in the radial direction have infinite value at point B. According to Guiggiani (1990), all boundary unknowns are still bounded and there is only a singularity in the contour derivative of the boundary displacement at corner B. Moreover, because there are no tractions on the internal edges of the ABC corner, the radial stress (tangential stress in the local coordinates) is obtained through the Hooke’s law from the tangential strain (differentiation of displacements), only. Thus, this stress component has one degree less in its interpolation than the element interpolation.

The edge BC of the internal corner was used to verify the smoothing scheme. Two meshes were employed: mesh 1, totaling 32 elements, and mesh 2 with 64 elements. 4 and 8 elements were used along edge BC, for each mesh, respectively. Additionally, a third, finer mesh (mesh 3) with a total of 268 elements, being 54 of them on edge BC, was used as a reference value. The three cases were analyzed with quadratic elements with \( P = 1 \). The offset used in all elements was 10% of the element lengths. Figure 8 shows the radial stress results obtained for meshes 1 and 2 up to half of the edge BC. The smoothing scheme generated satisfactory results from the half of the first element, with errors of less than 1.5%, as shown in Fig. 8a. However, the errors are considerably larger along the area closer to point B. Moreover, the raw solution is linear, while the smoothed solution is quadratic (same degree of the element).

In order to assess the influence of the mesh discretization, Fig. 8b shows the results for this case using a mesh with twice the number of elements (mesh 2). The result obtained show good agreement to the smoothed solution of mesh 3, except for \( r < 0.45 \).

![Figure 7. L-shaped domain.](image)

![Figure 8. Radial stress recovery along edge BC. Quadratic elements: (a) mesh 1, (b) mesh 2.](image)
4. CONCLUSIONS

This work introduced the application of SPR methods traditionally used in FEM context to obtain smoothed results in discontinuous boundary elements. More accurate interelement values were obtained in comparison to direct extrapolation of the original discontinuous solutions. The proposed technique performs efficiently to recover results in both types of discontinuous elements. Numerical results show that the smoothed solution converges as the mesh is refined. Moreover, the smoothed solutions converge faster than the standard BEM solutions.

Although dependence on other factors such as number of integration points, offset values, and mesh size should be further studied, the increase in the computational cost is negligible when compared to other processing stages.

The scheme proposed presents potential to be used with other types of discontinuous elements or different governing equations.

5. ACKNOWLEDGEMENTS

The authors are grateful for the financial support of CAPES, Brazil.

6. REFERENCES


7. RESPONSIBILITY NOTICE

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