AN ALTERNATIVE TECHNIQUE FOR TANGENTIAL STRESS CALCULATION IN DISCONTINUOUS BOUNDARY ELEMENTS

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Abstract. Continuous and discontinuous boundary elements obtain non-continuous and low accuracy results for the tangential component of stress, which is usually post-processed in the boundary element method (BEM). This paper presents a new proposal for calculation of this component for linear and quadratic discontinuous elements. The standard application of Hooke’s law uses functions with different degrees of interpolation, since the tangential component of deformation is obtained from the derivative of the interpolation functions of displacement. Therefore, the proposed technique is based on the use of a smaller number of points with higher convergence in the application of the Hooke’s law. The efficiency and capacity of the proposed technique is checked by solving static problems of elasticity with linear and quadratic elements, for different geometries and boundary conditions.

Keywords: Boundary elements method, discontinuous elements, stress recovery, tangential stress.

1. INTRODUCTION

The relaxation of continuity conditions in discretization-based methods has gained impulse among several numerical methods in the last decade. The use of discontinuous elements in boundary element methods (BEM) is somewhat old, but discontinuous Galerkin finite element methods (FEM) are prime examples of the developments in this field. These approaches greatly simplify the computational implementation of the solution methods, and may increase their efficiency, particularly in nonlinear problems or problems containing discontinuous fields.

In the BEM context, there are a number of advantages in the use of discontinuous boundary elements in spite of the characteristic interelement discontinuities. Discontinuous interpolation presents $C^1$ continuity on all physical nodes, which simplifies the computation of strongly singular integrals. It also avoids the need of double nodes in cases containing corners and discontinuities in the boundary conditions. In addition, the use of discontinuous elements has already proved its efficiency in the solution of multi-domain BEM formulations and FEM-BEM couplings (Zhang and Zhang, 2002). On the other hand, the recovery of variables at the ends of the elements by simple extrapolation or by averaging the extrapolated results of two or more elements is usually inadequate.

The issue approached by the present work is related to the boundary stress components not directly evaluated from BEM boundary solution. It is well known that in 2D elasticity problems only of two stress components are directly given by the traction components along the boundary. The remaining component of the stress tensor must be computed by mixing the known stress components and another term, evaluated by differentiating the shape functions in order to estimate the normal strain in the tangential direction. Regardless the elements are continuous or not, lower accuracy is generally found for these post-processed stress components due to the reduction by one degree in the approximation polynomial. Therefore, one can expect problems similar to those found in FEM for Mindlin plates, where the shear strain is evaluated by mixing p polynomials for the plate rotations with p-1 polynomials for the derivatives of the transverse displacement. This is not a robust method because of two reasons: (a) mixing primal variables with dual ones (obtained by numerical differentiation) may lead to ill-conditioned equations (Guiggiani, 1994); (b) nodes are not the optimal ordinates to recover derivative (dual) variables. Although it is a viable technique for many applications, the tangential stress component may present significant errors when coarse meshes are employed. Aiming the evaluation of more reliable values for the stress components on the boundary, a low-cost alternative technique for computing the normal tangential stress component is presented and tested in this work.

The proposed alternative technique for evaluation of the tangential stress is implemented for linear and quadratic discontinuous boundary elements, and used to solve 2-D elasticity benchmarks. The results obtained are compared with the conventional BEM results.

2. AN ALTERNATIVE TECHNIQUE FOR TANGENTIAL STRESS CALCULATION IN DISCONTINUOUS BOUNDARY ELEMENTS

In numerical analysis, the computation of quantities by combining interpolated values and its derivatives must be done with care, as the optimal sampling points of the derivatives are not coincident with the interpolation points themselves. This issue is relatively common in many branches of computational mechanics. Prime examples can be found in FEM, for instance, in the calculation of stress in two-dimensional elasticity elements, in the evaluation of shear strains in structural elements (plate/shell), or in the pressure-velocity coupling in fluid mechanics. This is essentially the
very same problem that causes the locking phenomenon of in low order thick plate finite elements (Oñate et al. 1992, Zienkiewicz et al. 1993). A similar problem occurs in the standard evaluation of the tangential boundary stress components for elasticity in BEM, although not characterized by the same consequences as in FEM.

As aforementioned, the missing boundary stress components in the conventional BEM are obtained using shape functions derivatives (tangential strain) and boundary tractions (Brebbia et al., 1984); however, it is known that this technique not necessarily provides good results along the whole element (Guiggiani, 1994). This work suggests a small change in the use of Hooke’s law in order to obtain a more reliable estimate of the tangential stress component for boundary elements without any significant increase in the computational cost. Basically, the tangential strain is sampled at optimal locations, instead of the nodes.

It is important to note that the ideas presented herein are implemented and tested for 2D elasticity discontinuous boundary elements, but they can be used in a fairly broad class of problems, regardless the continuity of the interpolation.

2.1. Standard technique for tangential stress calculation

In 2D elasticity problems, the normal (σnn) and shear (σnt) boundary stress are directly related to the boundary tractions (pn, pt) in a local coordinate system (n, t). Assuming that the tractions are written in the global coordinate system, the boundary stress components are easily obtained by rotating the tractions according to the local system:

\[
\begin{bmatrix}
\sigma_{nn} \\
\sigma_{nt}
\end{bmatrix} = \mathbf{R}(\alpha) \begin{bmatrix}
p_1 \\
p_2
\end{bmatrix}
\]

where \(\mathbf{R}(\alpha)\) is a rotation matrix and \(\alpha\) is the angle between global and the local coordinate systems (Fig. 1). The tangential strain \(\varepsilon_t\) is obtained by using the interpolated displacements (Zhao, 1996):

\[
\varepsilon_t = \frac{du_1}{dt} + \frac{du_2}{dt} t_1 + t_2 \quad \ldots \quad \varepsilon_t = \frac{1}{J} \sum \left( \frac{d\phi_1(\xi)}{d\xi} u'_1 t_1 + \frac{d\phi_1(\xi)}{d\xi} u'_2 t_2 \right)
\]

where \(t_1\) and \(t_2\) are components of the unit tangential vector in \(x_1\) and \(x_2\) directions, respectively, \(u'_1\) and \(u'_2\) are the nodal displacements in the \(i\)-th node in \(x_1\) and \(x_2\) directions, respectively; and \(\phi(\xi)\) are the physical interpolation functions \((J\) is the Jacobian of the element transformation to the normalized space).

![Figure 1. Coordinate system over the boundary.](image)

The tangential stress component \(\sigma_t\), can be obtained by Hooke’s law for plane-strain:

\[
\sigma_t(\xi) = \frac{1}{1+\nu} \left[ \nu \sigma_{nn}(\xi) + 2G\varepsilon_t(\xi) \right]
\]

where \(\nu\) is the Poisson’s ratio, and \(G\) is the shear modulus.

Therefore, when Eq.(3) is used in the standard BEM, it sums two polynomial terms of different orders. Depending on where \(\sigma_t\) is evaluated, this procedure process may lead to unreliable results unless the \(\xi\) coordinate is known to be an optimal point to retrieve derivative quantities (present in the \(\varepsilon_t\) term).
2.2. Alternative technique for tangential stress calculation

The existence of points able to represent optimally the derivative of an interpolated function is well known and can be proved mathematically. In the FEM, these points are known as Barlow points, and they are used to evaluate stress fields from differentiation of interpolated displacements (Barlow, 1976 and Prathap, 1996). When the interpolation function is of the polynomial type, these points are located at the Gauss-Legendre stations corresponding to one order less than the minimum order necessary to integrate the interpolation function exactly. The underlying idea of the scheme proposed here is to use these points to evaluate Eq.(3). To the best of the authors’ knowledge, there are no similar studies correlating these aspects in the BEM context.

In the case of linear boundary elements, the normal stress is obtained directly from the traction forces, and therefore it is a linear function (as well as the displacements). The tangential deformation is represented by a constant function in each element since it is obtained by the displacement derivative. The combined use of these two functions, through the Hooke’s law, is the origin of often unsatisfactory results (Guiggiani, 1994). The present work suggests the use of the central point of the element – Gauss point for a linear function integration ($\xi = 0$) – to sample the differentiation of the interpolated displacement. The coordinate $\xi = 0$ delivers the best estimate for the tangential strain along the element. It is worth to note that the evaluation of this strain at nodal locations ($\xi = \pm 1$) will overestimates or underestimates the strain value. In summary, it is proposed that both, the normal stress and tangential strain should be evaluated at the center of the element, thus obtaining a constant function for tangential stress over each element. For clarity, Table 1 compares both ways for the evaluation of the tangential stress in linear elements.

For quadratic discontinuous elements where, a priori, the normal stress and tangential strain are represented by quadratic and linear functions respectively, it is suggested that the Gauss points for a cubic quadrature ($\xi = \pm 1/\sqrt{3}$) should be used to represent the tangential strain field along the element. Therefore, replacing the standard technique, the calculation of the tangential stress is performed using the values for normal stress and tangential strain just at two points. A linear interpolation of the values obtained in the Gauss points is made in order to obtain the nodal stress values of the quadratic element. Table 2 shows the two methods for tangential stress calculation on quadratic elements.

In summary, the tangential stress calculation is made with one degree less than the other stress components. This method may initially seems less sound than the conventional procedure, but later it will be shown that when used with the smoothing technique described by Silveira (2007), the proposed scheme leads to better results. In many cases, it was found that the conventional scheme will produce wrong signs to the $\sigma_{tt}$ term in Eq.(3), a direct consequence of it being sampled at non-optimal points.

| Table 1. Tangential stress calculation for linear discontinuous boundary elements*. |
|---------------------------------|---------------------------------|
| STANDARD TECHNIQUE            | ALTERNATIVE TECHNIQUE          |
| $\sigma_n^i = \frac{1}{1-\nu}(\nu \sigma_{mn}^i + 2G\varepsilon_i^1)$ | $\sigma_n(\xi_i) = \frac{1}{1-\nu}(\nu \sigma_{mn}(\xi_i) + 2G\varepsilon_n(\xi_i))$ |
| $\sigma_n^2 = \frac{1}{1-\nu}(\nu \sigma_{mn}^2 + 2G\varepsilon_n^2)$ | $\sigma_n^i = \sigma_n^j$ refers to constant interpolation of $\sigma_n(\xi_i)$ |

* $\xi_i = 0$, and the superscript represent the associated nodal value of the variable.

| Table 2. Tangential stress calculation for quadratic discontinuous boundary elements*. |
|---------------------------------|---------------------------------|
| STANDARD TECHNIQUE            | ALTERNATIVE TECHNIQUE          |
| $\sigma_n^i = \frac{1}{1-\nu}(\nu \sigma_{mn}^i + 2G\varepsilon_i^1)$ | $\sigma_n(\xi_i) = \frac{1}{1-\nu}(\nu \sigma_{mn}(\xi_i) + 2G\varepsilon_n(\xi_i))$ |
| $\sigma_n^2 = \frac{1}{1-\nu}(\nu \sigma_{mn}^2 + 2G\varepsilon_n^2)$ | $\sigma_n(\xi_2) = \frac{1}{1-\nu}(\nu \sigma_{mn}(\xi_2) + 2G\varepsilon_n(\xi_2))$ |
| $\sigma_n^3 = \frac{1}{1-\nu}(\nu \sigma_{mn}^3 + 2G\varepsilon_n^3)$ | $\sigma_n^1, \sigma_n^2 \ e \sigma_n^3$ are obtained by linear interpolation of $\sigma_n(\xi_1) \ e \sigma_n(\xi_2)$ |

* $\xi_1 = -1/\sqrt{3}, \ \xi_2 = 1/\sqrt{3}$, and the superscript represent the associated nodal value of the variable.
3. NUMERICAL RESULTS

In order to investigate the performance of the proposed technique, this section shows some results. Numerical integration was carried out using 16 Gauss points, in order to minimize the influence of quadrature errors. Dimensions, material properties, and other physical data are given without units, but they were specified to represent a compatible system of units. The material properties used in all cases are: $E = 210\times10^9$ and $\nu = 0.3$. Plane-stress condition is assumed throughout this section.

3.1. Square-plate with a central hole under traction

A 100×100 square plate with a central hole of radius $R = 5$ was analyzed. Due to symmetry, only one quarter of the plate was considered (Fig. 2). The traction loading along the upper side was set to $P = 1$. The offset of all boundary elements used in the mesh is 15% of the element length.

Linear and quadratic elements were used with two different meshes for each type of element. Mesh 1 used an element size of 2.5 along the straight boundaries and four elements along the quarter-circle. Mesh 2 used an element size of 1.25 and eight elements along the quarter-circle.

Regarding the application of the alternative method for tangential stress calculation, it can be used with or without a smoothing procedure (Silveira, 2007), leading to four possibilities for post-processing the results:

- **Method A**: Discontinuous BEM without smoothing – the raw results of discontinuous elements are considered with standard tangential stress calculation (section 2.1).
- **Method B**: Discontinuous BEM with smoothing – same as Method A, but the results are smoothed.
- **Method C**: Modified discontinuous BEM without smoothing – raw results of discontinuous elements with alternative tangential stress calculation as outlined in section 2.2.
- **Method D**: Modified discontinuous BEM with smoothing – same as Method C, but the results are smoothed.

These methods were used to post-process the normal radial stress along the edge AB, which is the tangential component along that piece of boundary. Figure 3 compares graphically these results for linear elements with meshes 1 and 2.

The graphs depicted in Fig. 3 shows that the response which agrees more closely to the analytical solution is the smoothed solution considering the alternative tangential stress calculation. As expected, it can be seen that the alternative tangential solution without smoothing is simply an element average value from discontinuous BEM without smoothing.

Figure 4 shows the recovery of radial stress on the same edge, this time using quadratic elements. As in Fig. 3, these graphs show the four types of post-processed results against the analytical solution. Although the differences between the four methods are not as drastic as in the case of linear elements, it is evident that the smoothed results obtained with alternative tangential stress calculation agree more closely to the analytical solution, particularly at point A. Another important aspect is that the differences between all methods tend to vanish where analytical solution is less oscillatory (away from stress concentration areas).

Interestingly, it is also evident from Figs.3-4 that none of the methods provided very good results near the hole, although the modified stress calculation seems to recover the better ones. This is direct consequence of the different signals of the terms in Eq. (3), i.e. the high gradients of the tangential strain near the hole are miscalculated when the displacements are differentiated at non-optimal locations. Of course, this effect becomes more conspicuous when coarse meshes are used.
4. CONCLUSIONS

This work presented an alternative technique for tangential stress component calculation in BEM methods, which estimates more reliable results when compared to the standard boundary stress technique. Moreover, it was used a method to obtain smoothed results from the results of tangential stress component, since it is now evaluated at different positions than the nodes. This method of smoothing appears to work very well with the alternative technique for tangential stress calculation. The increase in the computational cost is negligible.

The scheme proposed presents potential to be used with other types of elements or different governing equations without hurdles.

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6. REFERENCES


7. RESPONSIBILITY NOTICE

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