INSTABILITY ANALYSIS USING A WAVELET-BASED MULTIRESOLUTION METHOD

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Abstract. The use of multiresolution techniques and wavelets has become increasingly popular in the development of numerical schemes for the solution of partial differential equations (PDEs), like the wavelet-Galerkin method. Therefore, the use of wavelets as basis functions in computational analysis holds some promise due to their compact support, orthogonality, localization and multiresolution properties, especially for problems with local high gradient, which would require a dense mesh in traditional methods, like the FEM. Another possible advantage is the fact that the calculation of the integrals of the inner products of wavelet basis functions and their derivatives can be made by solving a linear system of equations, thus avoiding the problem of approximating the integral by some numerical method. These inner products were defined as connection coefficients and they are employed in the calculation of stiffness, mass and geometry matrices. The formulation of a wavelet-based analysis is demonstrated for a one-dimensional problem using interpolating wavelets (Interpolets). Static and Instability examples are proposed.

Keywords: Wavelets, Interpolets, Multiresolution, Instability

1. INTRODUCTION

The use of wavelet-based numerical schemes has become increasingly popular in the last two decades. Wavelets have several properties that are especially useful for representing solutions of partial differential equations (PDEs), such as orthogonality, compact support and exact representation of polynomials of a certain degree. Their capability of representing data at different levels of resolution allow the efficient and stable calculation of functions with high gradients or singularities, which would require a dense mesh or higher order elements in a Finite Element analysis (Qian and Weiss, 1992).

A complete basis of wavelets can be generated through dilation and translation of a mother scaling function. Although many applications use only the wavelet filter coefficients of the multiresolution analysis, there are some which explicitly require the values of the basis functions and their derivatives, such as the Wavelet Finite Element Method (WFEM).

Compactly supported wavelets have a finite number of derivatives which can be highly oscillatory. This makes the numerical evaluation of integrals of their inner products difficult and unstable. Those integrals are called connection coefficients and they appear naturally when applying a numerical method for the solution of a PDE. Due to some properties of wavelet functions, these coefficients can be obtained by solving an eigenvalue problem using filter coefficients.

Working with dyadically refined grids, Deslauriers and Dubuc (1989) obtained a new family of wavelets with interpolating properties, later called Interpolets. Unlike Daubechies’ wavelets (Daubechies, 1988), Interpolets are symmetric, which is especially interesting in numerical analysis. The use of Interpolets instead of Daubechies’ wavelets considerably improves the method’s accuracy (Burgos et al., 2008).

The use of wavelets as interpolating functions in numerical schemes holds some promise due to their compact support, localization and multiresolution properties. The approximation of the solution can be improved by increasing either the level resolution or the order of the wavelet used.

Two examples were used for the validation of the proposed method. First, a beam with a concentrated load was used to test the method’s ability to capture singularities. In a second example, the critical loads and buckling modes of a doubly clamped beam were calculated at different levels of resolution.

2. WAVELET THEORY AND METHOD FORMULATION

2.1. Multiresolution Analysis

Multiresolution analysis using orthogonal, compactly supported wavelets has become increasingly popular in numerical simulation. Wavelets are localized in space, which allows the analysis of local variations of the problem at
various levels of resolution. In the following expression, known as the two-scale relation, $a_k$ are the filter coefficients of the wavelet scale function.

$$\varphi(x) = \sum_{k=-N}^{N-1} a_k \varphi(2x-k) = \sum_{k=-N}^{N-1} a_k \varphi_k(2x)$$

(1)

In general, there are no analytical expressions for wavelet functions. They can be obtained using iterative procedures like Eq. (1).

### 2.2. Interpolets

The basic characteristics of interpolating wavelets require that the mother scaling function satisfies the following condition (Shi et al., 1999):

$$\varphi(k) = \begin{cases} 1, & k = 0 \\ 0, & k \neq 0 \end{cases} \quad k \in \mathbb{Z}$$

(2)

The filter coefficients for Delauriers-Dubuc Interpolets can be obtained by an autocorrelation of the Daubechies filter coefficients. Interpolets satisfy the same requirements as other wavelets, specially the two-scale relation, which is fundamental for their use as interpolating functions in numerical methods. Figure 1 shows the Interpolet IN8. Its symmetry and interpolating properties are evident. There is only one integer abscissa which evaluates to a non-zero value.

![Figure 1. Interpolet IN8 scaling function with its full support](image)

### 2.3. Connection Coefficients

Assuming that a function $f$ is approximated by a series of interpolating scale functions, the following may be written:

$$f(\xi) = \sum_i a_i \varphi_i(\xi)$$

(3)

The process of solving a differential equation by some numerical method requires the calculation of the inner products of the basis functions and their derivatives. These inner products are defined as connection coefficients:

$$\Lambda_{ij}^{d \times d} = \int \varphi^{(d)}(\xi-i) \varphi^{(d)}(\xi-j) \, d\xi$$

(4)

The values for the limits of the integral in Eq. (4) depend on which method is used to impose boundary conditions. In this work, the limits are given by $[0, 2^m]$, where $m$ is the wavelet level of resolution. This method allows the use of Lagrange multipliers to deal with boundary conditions, similarly to what is usually done in a meshless scheme (Nguyen et al., 2008) Connection coefficients at level $m$ can be obtained through the calculation at level 0 thus avoiding its
recalculation while increasing the level of resolution. Wavelet dilation and translation properties allow the calculation of connection coefficients within the interval [0 1] to be summarized by the solution of an eigenvalue problem based only on filter coefficients (Zhou & Zhang, 1998).

\[
\left( P - \frac{1}{2^d} \right) A_{d_1} = \mathbf{0}
\]  

(5)

\[
P = \left[ a_{r_2} a_{r_2} + \ldots \right]_{i,j,r,s \in \{2\ldots N\}}
\]  

(6)

where \( N \) is the wavelet order. Since Eq. (5) leads to an infinite number of solutions, there is the need for a normalization rule that provides a unique eigenvector. This unique solution comes with the inclusion of the so-called moment equation, derived from the wavelet property of exact polynomial representation (Latto et al, 1992).

\[
\sum_{i} \sum_{j} M^i M^j A_{d_{i,j}} = \frac{(k!)^2}{(k-d_1)(k-d_2)(2k-d_1-d_2+1)}
\]  

(7)

\[
M' = \frac{1}{2^d} \sum_{k=0}^{j} \left( \sum_{l=0}^{i} \sum_{\xi=0}^{j} \right) \left( \sum_{\xi=0}^{j} \right) \left( \sum_{\xi=0}^{j} \right)
\]  

(8)

2.4. Applications

The numerical solution of differential equations is one of the possible applications of the wavelet theory. The equation of a beam subjected to an axial load is given by:

\[
\frac{\partial^4 w}{\partial x^4} + \frac{P}{E I} \frac{\partial^2 w}{\partial x^2} = 0
\]  

(9)

Stiffness and geometry matrices can be obtained by substituting the displacement \( w \) by a series of interpolating functions. Adimensional coordinates \( \xi \) within the interval [0,1] are used in wavelet space, which leads to the subsequent expressions:

\[
\tilde{k}_{i,j} = \int_{0}^{1} \phi_i(\xi) \phi_j^*(\xi) d\xi = \Lambda_{i,j}^{2,2}
\]  

(10)

\[
\tilde{g}_{i,j} = \int_{0}^{1} \phi_i(\xi) \phi_j^*(\xi) d\xi = \Lambda_{i,j}^{1,1}
\]  

(11)

As done in a FE scheme, the critical loads and buckling modes can be obtained by solving an eigenvalue problem of the form:

\[
\begin{bmatrix}
\Lambda_{2,2} & -G \\
-G^T & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{a}
\end{bmatrix}
= \frac{P}{E I}
\begin{bmatrix}
\Lambda_{1,1} & 0 \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\tilde{\lambda}
\end{bmatrix}
\begin{bmatrix}
0
\end{bmatrix}
\]  

(12)

where \( G \) is a matrix associated with boundary conditions and \( \lambda \) is a vector of Lagrange multipliers. The main difference in relation to the FEM is that the unknowns in Eq. (12) are the interpolating coefficients of the basis functions instead of nodal displacements. In fact, there is no need to establish nodal coordinates.

3. EXAMPLES

Figure 2 shows a simple example of a beam subjected to a concentrated load at its midpoint. This example was formulated in order to verify the ability of the wavelet method to deal with singularities, since the load generates a discontinuity in the shear force diagram.
This example is easily solved by dividing the beam in two elements and applying the load as a nodal force. In this work, since degrees of freedom don’t have a fixed position, the load is transformed into the wavelet space:

\[
q(\xi) = P\delta\left(\xi - \frac{1}{2}\right) \rightarrow \int_0^1 q(\xi)\varphi(\xi - i)d\xi = P\int_0^1 \delta\left(\xi - \frac{1}{2}\right)\varphi(\xi - i)d\xi = P\varphi\left(\frac{1}{2} - i\right)
\]

(13)

The example was solved using IN8 Interpolet at different levels of resolution and the results for bending moment and shear force diagrams are showed in Figs. 3 and 4.
It is clear that higher levels of resolution are necessary in order to capture the singularity that appears where the load is applied. Nevertheless, results are considerably good, considering that the solution is obtained in wavelet space and no discretization was performed. The discontinuity in the slope of the bending moment is captured even for a low level of resolution.

In a second example, critical loads for a doubly clamped beam were obtained by solving an eigenvalue problem using stiffness and geometry matrices, as in Eq. (12). Results at different levels of resolution are showed in Tab. (1).

Table 1. Critical loads obtained at different levels of resolution for a double clamped beam.

<table>
<thead>
<tr>
<th>MODE Nº.</th>
<th>EXACT</th>
<th>LEVEL 0</th>
<th>LEVEL 2</th>
<th>LEVEL 4</th>
<th>LEVEL 6</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>80.7629</td>
<td>81.7817</td>
<td>80.7779</td>
<td>80.7629</td>
<td>80.7629</td>
</tr>
<tr>
<td>3</td>
<td>157.9137</td>
<td>159.1386</td>
<td>158.6607</td>
<td>157.9145</td>
<td>157.9137</td>
</tr>
</tbody>
</table>

Figure 5. Buckling modes for example 2

Figure 6. Relative error in third critical load at different levels of resolution
Figure 5 shows the shape of the first three buckling modes obtained for the mentioned example. Results were normalized. The same example was analyzed using IN6 Interpolet in order to verify the influence of wavelet order. Figure 6 shows the relative error obtained for the third critical load at different levels of resolution for each Interpolet.

4. CONCLUSIONS

This work presented the formulation and validation of a wavelet-based method for obtaining critical loads and buckling modes for axially loaded beams. It was also shown that wavelets have the ability of capturing discontinuities in the solution of differential equations without the need to create a node at the point where the singularity occurs.

As in the traditional FEM and other numerical methods, the accuracy of the solution can be improved either by increasing the level of resolution or the wavelet order. Sometimes, lower order wavelets at higher resolution can give better results than higher order wavelets at lower resolutions.

All matrices involved can be stored and operated in a sparse form, since most of their components are null, thus saving computer resources. Due to the compact support of wavelets, the sparseness of matrices increases along with the level of resolution.

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6. REFERENCES


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