ANALYSIS OF A CRACKED FUNCTIONALLY GRADED STRIP WITH ARBITRARY DISTRIBUTED ELASTIC MODULUS

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Abstract. The plane strain deformation for a crack in a functionally graded strip with arbitrary variations of material properties is studied by a newly formulated multi-layered model. The governing equation in terms of Airy stress function is solved by means of Fourier transformation. The mixed boundary problem is reduced to a system of singular integral equations that are solved numerically to obtain the stress intensity factors at crack tips. The strain energy density criterion is employed to predict the direction of crack initiation. A numerical example is given to investigate the influence of crack sizes and graded material properties on the fracture behavior of functionally graded materials.

Keywords: functionally graded materials, crack, plane deformation, stress intensity factor, Fourier transformation

1. INTRODUCTION

Functionally graded materials (FGMs) have been developed in many application fields to eliminate stress concentrations, relax residual stresses, and enhance bonding strength of materials. It has been experimentally observed that crack propagation in FGMs is the most common failure mode of a metal-ceramic FGM when it is subjected to loads. Therefore, a lot of efforts have been devoted to the study of fracture behavior of FGMs (e.g., Erdogan, 1995; Jin and Batra, 1996; Gu and Asaro, 1997; Rousseau and Tippur, 2000).

The main features of crack-tip stress field in a functionally graded material lie in that the crack tip singularity is identical to that of a homogeneous material and hence the stress intensity factors (SIFs) can still be defined and used as a fracture parameter for FGMs. However, due to mathematical difficulties arising from the continuously varying material properties, the stress intensity factors that can be analytically obtained are rather limited. Most of the available works assumed the elastic modulus varying in an exponential form from which the governing differential equations were reduced to those with constant coefficients and exact solutions could be easily found (Erdogan, 1995; Jin and Batra, 1996). As for other variations of elastic moduli, the governing differential equations for FGMs are of variable coefficients and exact solutions are difficult to obtain. Various approximate models have been proposed to capture arbitrarily varying material properties of functionally graded materials (Wang et. al., 2002; Huang et. al., 2003; Guo and Noda, 2007).

In the present paper, a multi-layered model is employed for the analysis of the crack problems in FGMs, which models arbitrary variations of material properties based on two linear-distributed material parameters. In Section 2, problem formulation is given; In Section 3, solution by Fourier transformation is obtained; In Section 4, crack initiation criterion based on the strain energy density criterion is described; Finally, in Section 5, a numerical example is given and discussion is made.

2. PROBLEM FORMULATION

Consider a FGM strip that is infinite along x-axis and has a finite thickness \( h \). The strip contains a through crack of length \( 2c \) that is parallel to the edges of the strip, as shown in Fig. 1. The constitutive equations of FGMs under plane strain deformation are given as

\[
\varepsilon_{xx} = a\sigma_{xx} - b\sigma_{yy}, \quad \varepsilon_{xy} = a\sigma_{xy} - b\sigma_{xx}, \quad \gamma_{xy} = 2(a+b)\sigma_{xy}
\]

where \( a \) and \( b \) are two material parameters which are related to Young’s modulus \( E \), shear modulus \( \mu \) and Possion’s ratio \( \nu \), as follows:

\[
a = \frac{1-\nu^2}{E} = \frac{1-\nu}{2\mu}, \quad b = \frac{\nu(1+\nu)}{E} = \frac{\nu}{2\mu}
\]

For a functionally graded material, \( a \) and \( b \) are functions of coordinates and in the present study, they are assumed to be functions of coordinate \( y \), i.e., \( a = a(y) \), \( b = b(y) \).

In order to simulate the arbitrary variations of \( a(y) \) and \( b(y) \), a multi-layered model can be employed. This multi-layered model is based on the fact that an arbitrary curve can be approximated by a series of continuous but
piecewise linear curves. The FGM strip is divided into \( L \) sub-layers with the crack on the \( K \)th sub-interface (\( K \) may be any integer between 1 and \( L \)), as shown in Fig. 2. The material parameters \( a(y) \) and \( b(y) \) are assumed to vary linearly in each sub-layer and is continuous at the interfaces of sub-layers, i.e.,

\[
a(y) = \overline{a}_j \left( 1 + \alpha_j \frac{y}{h_j} \right) \quad b(y) = \overline{b}_j \left( 1 + \beta_j \frac{y}{h_j} \right) \quad (h_{j-1} < y < h_j, J = 1, 2, \ldots, L) \quad (3)
\]

![Figure 1 A FGM strip with a crack](image1.png)

![Figure 2 A multi-layered model](image2.png)

where

\[
\alpha_j = \frac{h_j a(h_{j-1}) - h_{j-1} a(h_j)}{h_j - h_{j-1}} \quad \beta_j = \frac{h_j b(h_{j-1}) - h_{j-1} b(h_j)}{h_j - h_{j-1}} \quad (4)
\]

In every sub-layer, the governing equation is obtained as

\[
\nabla^2 \nabla^2 F_j + \frac{2\alpha_j}{h + \alpha_j y} \frac{\partial}{\partial y} (\nabla^2 F_j) = 0 \quad (J = 1, 2, \ldots, L) \quad (5)
\]

where \( F_j \) is Airy stress function in \( J \)th sub-layer, and \( \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \) is two dimensional Laplace operator.

The continuous conditions of the displacements and stresses at the interfaces of sub-layers (\( y = h_j \), \( J = 1, 2, \ldots, L-1 \) and \( J \neq k \)) are written as

\[
\sigma_{yJ} (x, h_J) - \sigma_{y(J+1)} (x, h_J) = 0 \quad \sigma_{xJ} (x, h_J) - \sigma_{x(J+1)} (x, h_J) = 0 \quad (6)
\]

\[
u_{xJ} (x, h_J) - \nu_{x(J+1)} (x, h_J) = 0 \quad \nu_{yJ} (x, h_J) - \nu_{y(J+1)} (x, h_J) = 0 \quad (7)
\]

The boundary conditions on the crack plane are given as

\[
\sigma_{yK} (x, h_K) = -\sigma(x) \quad \sigma_{xK} (x, h_K) = -\tau(x) \quad |x| \leq c \quad (8)
\]

\[
u_{xK} (x, h_K) - \nu_{x(K+1)} (x, h_K) = 0 \quad \nu_{yK} (x, h_K) - \nu_{y(K+1)} (x, h_K) = 0 \quad |x| > c \quad (9)
\]

\[
\sigma_{yK} (x, h_K) - \sigma_{y(K+1)} (x, h_K) = 0 \quad \sigma_{yK} (x, h_K) - \sigma_{y(K+1)} (x, h_K) = 0 \quad |x| > c \quad (10)
\]

The upper and lower surfaces of the strip are free of traction, i.e.

\[
\sigma_{y0} (x, 0) = \sigma_{yL} (x, 0) = \sigma_{yL} (x, h) = \sigma_{yL} (x, h) = 0 \quad (11)
\]

Now the problem is reduced to solve the governing equation (5) under boundary conditions (6)-(11).
3. SOLUTION BY FOURIER TRANSFORMATION

Applying Fourier transformation with respect to \( x \), Eq. (5) is transformed to

\[
\frac{d^4 \tilde{F}_j}{dy^4} + \frac{2\alpha_j}{h + \alpha_j y} \frac{d^3 \tilde{F}_j}{dy^3} - 2s^2 \frac{d^2 \tilde{F}_j}{dy^2} - \frac{2\alpha_j}{h + \alpha_j y} s^2 \frac{d \tilde{F}_j}{dy} + s^4 \tilde{F}_j = 0
\]  (12)

where \( \tilde{F}_j = \tilde{F}_j(s, y) = \int_{-\infty}^{\infty} \tilde{F}_j(x, y)e^{-i2\pi s} dx \) is the Fourier transform of the Airy stress function. The general solution of Eq. (12) is written as

\[
\tilde{F}_j = A_{1j}(s) f_1(j, s, y) + A_{2j}(s) f_2(j, s, y) + A_{3j}(s) f_3(j, s, y) + A_{4j}(s) f_4(j, s, y)
\]  (13)

where \( f_1(s, y), f_2(s, y), f_3(s, y), f_4(s, y) \) are four independent particular solutions:

\[
\begin{align*}
f_1(s, y) &= \exp(t_j) \\
f_2(s, y) &= \exp(-t_j) \\
f_3(s, y) &= \exp(t_j)Ei(-2t_j) - \exp(-t_j) \ln(t_j) \\
f_4(s, y) &= \exp(-t_j)Ei(2t_j) - \exp(t_j) \ln(t_j)
\end{align*}
\]  (14)

and \( t_j = \left| s \right| \left( y + \frac{h}{\alpha_j} \right) \), \( Ei(t) = \int_{t}^{\infty} e^{-\zeta} d\zeta = \int_{-\infty}^{\zeta} e^{\zeta} d\zeta \) is the exponential integration function.

\( A_{1j}(s), A_{2j}(s), A_{3j}(s), A_{4j}(s) \) are unknown functions of \( s \) determined from boundary conditions (8)-(11) which has been given in detail in Zhong and Cheng (2008). Accordingly, the Fourier transforms of displacements and stresses in \( J^{th} \) sub-layer can be obtained as

\[
\begin{align*}
\tilde{u}_{xj}(s, y) &= -i \left( \frac{a}{s} \frac{d^2 \tilde{F}_j}{dy^2} + bs^2 \tilde{F}_j \right) \\
\tilde{u}_{yj}(s, y) &= \frac{a}{s^2} \frac{d^2 \tilde{F}_j}{dy^2} + \frac{1}{s} \frac{da}{dy} \frac{d \tilde{F}_j}{dy} - (2a + b) \frac{d \tilde{F}_j}{dy} + db \tilde{F}_j \\
\tilde{\sigma}_{xxj}(s, y) &= \frac{d^2 \tilde{F}_j}{dy^2} \\
\tilde{\sigma}_{yj}(s, y) &= -s^2 \tilde{F}_j \\
\tilde{\sigma}_{yxj}(s, y) &= -is \frac{d \tilde{F}_j}{dy}
\end{align*}
\]  (15)

Conducting the inverse Fourier transformation we can obtain the displacement and stress components and accordingly, the stress intensity factors at the crack tips are defined as

\[
K_i^{\pm} = \lim_{s \to \pm c} \sqrt{2\pi} c \sigma_{xj}(x, h_k) = \mp \frac{1}{4a(h_k)} \sqrt{c} \phi_2(\pm c)
\]  (16)

\[
K_{II}^{\pm} = \lim_{s \to \pm c} \sqrt{2\pi} c \sigma_{yj}(x, h_k) = \mp \frac{1}{4a(h_k)} \sqrt{c} \phi_1(\pm c)
\]  (17)

4. CRACK INITIATION CRITERION

In order to predict the direction of crack initiation in a functionally graded material with varying properties, whose stress state ahead of the crack is often mixed, the strain energy density criterion (Sih and Macdonald, 1974) will be employed. The corresponding strain energy density factor \( S \) near the crack tip is written as
\[ S = a_{11} K_1^2 + a_{22} K_\Pi^2 + 2a_{12} K_1 K_\Pi \] (18)

with
\[
a_{11} = \frac{1}{16\pi\mu(h_k)} [3 - 4v(h_k) - \cos\theta][1 + \cos\theta] \\
a_{12} = \frac{1}{16\pi\mu(h_k)} 2\sin\theta \{ \cos\theta - [1 - 2v(h_k)] \} \\
a_{22} = \frac{1}{16\pi\mu(h_k)} [4(1 - v(h_k))[1 - \cos\theta] + (1 + \cos\theta)(3\cos\theta - 1)]
\]

where \( \mu(h_k) \) and \( \nu(h_k) \) are the shear modulus and Possion’s ratio at the crack tip, respectively. The strain energy density fracture criterion states that:

1. Crack initiation takes place in a direction \( \theta = \theta_0 \) of the minimum strain energy density, i.e.,
\[
\frac{\partial S}{\partial \theta} = 0, \quad \frac{\partial^2 S}{\partial \theta^2} > 0, \quad \text{at} \quad \theta = \theta_0
\] (19)

2. Crack extension occurs when the minimum strain energy density factor at the direction \( \theta = \theta_0 \) reaches a critical value \( S_c \), i.e.,
\[
S_{\text{min}} = S_c
\] (20)

5. NUMERICAL RESULTS AND DISCUSSION

As a demonstrating example, we consider a FGM strip with a midline crack (i.e., \( h_k = h/2 \)) of length 2\( c \) subjected to a uniform tensile loading \( \sigma(x) = \sigma_0 \). The geometry of the problem being examined is shown in Fig. 1. Assume that Possion’s ratio takes a constant \( \nu = 0.3 \) and the shear modulus is given in the following three forms (as shown in Fig. 3):

1. \[
\frac{\mu(y)}{\mu_0} = \left( \frac{\mu_h}{\mu_0} \right)^{y/h} \quad \text{(Exponential function model)}
\]

2. \[
\frac{\mu(y)}{\mu_0} = 1 + \left( \frac{\mu_h}{\mu_0} - 1 \right) \left( \frac{y}{h} \right)^2 \quad \text{(Parabolic function model)}
\]

3. \[
\frac{\mu(y)}{\mu_0} = 1 + \left( \frac{\mu_k}{\mu_0} - 1 \right) \sin \left( \frac{\pi y}{2h} \right) \quad \text{(Sinusoidal function model)}
\]

where \( \mu_0 = \mu(0) \) and \( \mu_k = \mu(h) \). The multi-layered model proposed in Section 2 is employed to study this crack problem. In order to obtain enough accurate results and to avoid too much CPU time, we need to choose properly total number of the discrete points (i.e., \( N \) in Eqs. (34) and (35)) and the total number of sub-layers (i.e., \( L \) in Eq. (3)). It is found from detail numerical tests that sufficiently accurate results can be obtained when \( N \geq 50 \) and \( L \geq 6 \) for the present numerical example.

Figure 4 depicts the variation of the normalized Mode I and Mode II stress intensity factors (SIFs), \( K_I/(\sigma_0 \sqrt{c}) \) (Fig. 4(a)) and \( K_\Pi/(\sigma_0 \sqrt{c}) \) (Fig. 4(b)), with the normalized crack length \( c/h \) for above three material graded models under uniform normal loading at the crack-face \( \sigma(x) = \sigma_0 \) \((-c < x < c)\) when \( \mu_0/\mu_h = 10 \). Since SIFs for homogeneous materials do not depend on shear modulus, the corresponding SIFs for homogeneous materials are also given in Fig. 4. It can be found from Fig. 4 that Mode I stress intensity factors of functionally graded materials reduces considerably compared to those of homogeneous materials at the cost of inducing Mode II stress intensity factors. As a result, a mixed fracture mode should be considered.
Figure 5 shows the dependence of the critical kink angle (denoted by $-\theta_0$), describing the direction of crack initiation, on the normalized crack length $c/h$ when $\mu_0/\mu_h = 10$ (see Fig. 5(a)) and the inhomogeneous parameter $\mu_0/\mu_h$ when $c/h = 1$ (see Fig. 5(b)) under uniform normal loading at the crack-face $\sigma(x) = \sigma_0$ ($-c < x < c$). It can be seen from Fig. 5(a) that the crack kink angle ($-\theta_0$) decreases with the increase of the normalized crack length $c/h$ for above three material graded models. This observation indicates that a bigger crack tends to propagate along the direction near the original crack line while a smaller crack is much easier to extend deviating from the original crack line. It is also found from Fig. 5(b) that the inhomogeneous parameter $\mu_0/\mu_h$ greatly influences the critical kink angle. The critical kink angle vanishes when $\mu_0/\mu_h = 1$ (corresponding to homogeneous materials) but is greater than $20^\circ$ when $\mu_0/\mu_h = 20$ for above three material graded models. However, for different material graded models with the same inhomogeneous parameter $\mu_0/\mu_h$, the difference of the critical kink angle is not very big. For example, the difference of the critical kink angle for above three material graded models is below $5^\circ$.

The following conclusions can be made from above observations: 1) fracture toughness of materials can be greatly changed by introducing a graded variation of elastic modulus; 2) the influence of the specific form of elastic modulus on the fracture behavior of a FGM does exist but is not very big. These conclusions may be helpful in designing and manufacturing functionally graded materials.

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7. REFERENCES


Figure 4 Variations of (a) the normalized stress intensity factors $K_I/(\sigma_0 \sqrt{c})$ and (b) $K_{II}/(\sigma_0 \sqrt{c})$ with $c/h$ under uniform normal loading at the crack-face $\sigma(x) = \sigma_0 \ (c < x < c)$ for different material graded models.

Figure 5 Variation of the critical kink angle with (a) the normalized crack length $c/h$ when $\mu_0/\mu_h = 10$ and (b) the inhomogeneous parameter $\mu_0/\mu_h$ when $c/h = 1$ under uniform normal loading at the crack-face $\sigma(x) = \sigma_0 \ (c < x < c)$.

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