ANALYTICAL PROCEDURE FOR FREE VIBRATION ANALYSIS OF BEAM-CAVITY SYSTEMS

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Abstract. This paper presents an analytical procedure for solution of the dynamic interaction problem between a simple beam and a rectangular cavity containing an acoustic fluid. Initially an analytical expression for the fluid domain with a harmonic vibrating boundary is established, based on the separation of variables technique. This solution provides the dynamic pressure field, which is dependent of two unknowns: the coupled system frequency and the vibrating boundary deformed shape. The dynamic equilibrium equation of the coupled structure is defined using the virtual work principle, with the dynamic pressures acting as external forces. Solution of this expression is achieved upon the definition of an imposed deformation function for the structure, providing the coupled frequencies for a given mode shape. These results can be readily applied in the analytical expression for the fluid domain, providing the correspondent cavity modes. Comparisons of this procedure with Finite Element Method models indicate a good agreement between both solutions. Some advantages of this technique include the possibility of parametric analysis and the validation of numerical solutions.

Keywords: analytical methods, fluid-structure, structural dynamics, free vibration, acoustic cavities

1. INTRODUCTION

The fluid-structure interaction problem involves the determination of structure and fluid responses. However, it is important to emphasize that these responses are not independent. Fluid domain pressure field depends on the structural displacement, which in turn depends on the forces exerted by the fluid at the interface. Analytical solutions for the free vibration problem of an acoustic cavity backed by a panel were initially studied by Dowell and Voss (1963), and Pretlove (1965). Solutions that consider water interaction effects in a dam excited by an earthquake or in a vibrating column are also available in literature (Westergaard, 1933; Chopra, 1970; Zhu et al., 1989; Ribeiro et al., 2009). The pressure field produced by these solutions along the fluid-structure interface represents exactly the loads that must be added to the structure, providing the coupling effects between these two systems. Analytical eigenvalues expressions for this problem are also available in literature (Xing et al., 1997), however solutions are often quite involving.

The studies presented below involve the free vibration analysis of beam-cavity systems in generalized coordinates, considering an imposed deformation function for the structure. The cavity is assumed filled with an acoustic fluid, governed by the two dimensional wave equation. A simplified analytical procedure is proposed, where the coupled frequencies for a given structural mode shape is obtained upon the solution of a frequency equation, where the fluid effects act as an added mass. The coupled frequencies can be readily applied in the analytical solution for the fluid domain, providing the coupled cavity modes.

2. ANALYTICAL SOLUTION FOR THE FLUID DOMAIN

The basic assumptions of this problem are the corresponding to treatment of this medium as an acoustic fluid. With these considerations, it is assumed that the fluid transmits only pressure waves. Some applications of this theory include: propagation of pressure waves in pipes and sound waves propagating through fluid-solid media. The following basic assumptions are made for the solution of this problem (based on the propositions of Chopra, 1970, Rashed 1983 and Silva, 2007): a. homogeneous, inviscid and linearly compressible fluid; b. irrotational flow; c. displacements and their derivatives are small; d. surface wave effects are neglected; e. the fluid-structure interface movement is bidimensional (the same for any vertical plane perpendicular to the structural axis); f. the fluid-structure interface is vertical; g. interface displacements are represented by an arbitrary deformation function.

The previous assumptions lead to a dynamic pressure distribution \( p(x, y, t) \), in excess of the static pressure, given by:

\[
\nabla^2 p = \frac{1}{c^2} \frac{\partial^2 p}{\partial t^2}
\]

which corresponds to the bidimensional wave equation, with \( c \) representing the fluid sonic velocity. Solution of this last expression is achieved using the separation of variables technique. Therefore it is assumed that this expression can be separated, resulting in:
\[ p(x, y, t) = F(x)G(y)T(t) \]  

A simple and time independent solution for this problem can be achieved with the hypothesis of time harmonic vibrations, with frequency \( \omega \). Thus it is assumed that the time related function is given by:

\[ T(t) = e^{-i\omega t} \]

Substitution of Eq. (3) in Eq. (2) provides:

\[ p(x, y, t) = F(x)G(y) e^{-i\omega t} \]

which can be readily substituted in Eq. (1), resulting in:

\[ \nabla^2 P + \left( \frac{\omega}{c} \right)^2 P = 0 \]  

with \( P(x, y) = F(x)G(y) \). This last expression represents Helmholtz’s classical differential equation, or the reduced wave equation, which is time independent. Solution of Eq. (5) is established upon the solution of two ordinary differential equations involving the longitudinal \( x \) and the transversal \( y \) directions. Thus:

\[
\frac{F''(x)}{F(x)} + \frac{G''(y)}{G(y)} = -\left( \frac{\omega}{c} \right)^2 \quad \therefore -\frac{G''(y)}{G(y)} = \beta \quad ; \quad \frac{F''(x)}{F(x)} = \beta - \left( \frac{\omega}{c} \right)^2
\]

where the tiles indicate derivatives related to the correspondent variable. A total of four boundary conditions are needed (two for each direction), and these are defined by the analyzed cavity model (illustrated on Fig. 1). Application examples are provided below.

**2.1. Solution for an open cavity with a vibrating boundary (Case 1)**

In this case it is assumed that boundaries \( S_2 \), \( S_3 \) and \( S_4 \) are open. This implies in a null pressure at these locations. Thus, the contour conditions for this problem are defined by:

\[
S_1 \rightarrow \frac{\partial p}{\partial x} \bigg|_{x=0} = -\rho \beta \bar{u} \quad ; \quad S_2 \rightarrow p(L_x, y, t) = 0 \tag{7}
\]

\[
S_3 \rightarrow p(x, 0, t) = 0 \quad ; \quad S_4 \rightarrow p(x, L_y, t) = 0 \tag{8}
\]

The previous listed equations define the four boundary conditions needed for solution of Eq. (5). Thus the correspondent expression for \( P(x, y) \) is given by:
\[ P(x, y) = -\frac{2\rho_f A}{L_y} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\alpha_n}} \int_0^L \phi(y) \sin(\kappa_n y) \, dy \]
\[ \cdot \left[ \sin\left(\sqrt{\alpha_n} x\right) - \tan\left(\sqrt{\alpha_n} L_x\right) \cos\left(\sqrt{\alpha_n} x\right) \right] \sin(\kappa_n y) \]

where:
\[ \kappa_n = \frac{n\pi}{L_y} ; \quad \alpha_n = \left(\frac{\omega}{c}\right)^2 - \kappa_n^2 \]  

2.2. Solution for a closed cavity in the longitudinal direction with a vibrating boundary (Case 2)

In this case it is assumed that boundaries \( S_3 \) and \( S_4 \) are open. Additionally it is assumed that the cavity is closed at \( S_2 \) boundary. Thus, the contour conditions for this problem are defined by:

\[ S_1 \rightarrow \frac{\partial p}{\partial x} \bigg|_{x=0} = -\rho_f \ddot{u} \quad ; \quad S_2 \rightarrow \frac{\partial p}{\partial x} \bigg|_{x=L_x} = 0 \]
\[ S_3 \rightarrow p(x, 0, t) = 0 \quad ; \quad S_4 \rightarrow p(x, L_y, t) = 0 \]

Thus the correspondent expression for \( P(x, y) \) is given by:

\[ P(x, y) = -\frac{2\rho_f A}{L_y} \sum_{n=1}^{\infty} \frac{1}{\sqrt{\alpha_n}} \int_0^L \phi(y) \sin(\kappa_n y) \, dy \]
\[ \cdot \left[ \cos\left(\sqrt{\alpha_n} x\right) + \tan\left(\sqrt{\alpha_n} L_x\right) \sin\left(\sqrt{\alpha_n} x\right) \right] \sin(\kappa_n y) \]

where \( \kappa_n \) and \( \alpha_n \) are given by Eq. (10).

3. GENERALIZED COORDINATE SOLUTION FOR THE COUPLED SYSTEM

The analyzed problem is illustrated on Fig. 2. It consists on a general structure with a constant cross-section subjected to an external load. The dynamic response of this system can be represented by a generalized coordinate \( X(t) \), allowing the construction of generalized parameters (mass, stiffness and loading) for any arbitrary mode shape. This type of solution will be very useful for the introduction of fluid pressures, since the previous developed approach for the fluid domain is also dependent on the vibrating boundary deformation \( \phi(y) \).

Figure 2. Representation scheme of the structural model
For the mathematical development of this problem the following considerations are assumed: mass per unit length \( \mu(y) \), flexural rigidity \( EI(y) \), length \( L_y \), external distributed loading \( F(y,t) \) and unitary width perpendicular to the \( xy \) plane. The deflections are represented by \( \nu(y,t) \), related to an arbitrary coordinate \( X(t) \) and a mode shape function \( \phi(y) \), normalized at the generalized coordinate location. The system’s dynamic equilibrium equation is obtained with the virtual work principle, equaling the work done by internal and external forces. Thus:

\[
\ddot{M} \ddot{X} + \dddot{K} \dot{X} = \dddot{F}(t) \tag{14}
\]

where:

\[
\dddot{M} = \int_0^{L_y} \mu(y) \left[ \phi(y) \right]^2 dy \tag{15}
\]

\[
\dddot{K} = \int_0^{L_y} EI(y) \left[ \frac{d^2 \phi(y)}{dy^2} \right]^2 dy \tag{16}
\]

\[
\dddot{F} = \int_0^{L_y} F(y,t) \phi(y) dy \tag{17}
\]

### 3.1. Application of fluid pressures at the interface

The previous solution depends on an external loading. In the case of a coupled system this loading is represented by the dynamic pressures acting at the interface. Therefore:

\[
F(y,t) = p(0,y,t) \times 1 = P(0,y) e^{-iot} \tag{18}
\]

where the function \( P(0,y) \) is related to the corresponding cavity (associated to the framed structure), with boundary conditions depending on the analyzed solution for the fluid domain. For a harmonic vibrating contour, accelerations at the interface are given by:

\[
\dddot{u}(y,t) = \phi(y) \dddot{A} \ e^{-iot} \tag{19}
\]

Accelerations at the interface are equivalent for both the structure and the fluid. Therefore:

\[
\dddot{u}(y,t) = \dddot{v}(y,t) = \dddot{\phi}(y) \dddot{X}(t) \tag{20}
\]

Analysis of Eq. (19) and Eq. (20) provides:

\[
\dddot{X}(t) = \dddot{A} \ e^{-iot} \tag{21}
\]

Substitution of (17), (18) and (21) in (14) gives:

\[
\left[ \dddot{M} + \int_0^{L_y} \frac{P(0,y)}{\dddot{A}} \phi(y) dy \right] \dddot{X} + \dddot{K} \dddot{X} = \left[ \dddot{M} + \dddot{M}_\text{fluid} \right] \dddot{X} + \dddot{K} \dddot{X} = 0 \tag{22}
\]

In the above expression the generalized force is located at the left hand side, because physically the dynamic pressure acts in the same direction of inertia and elastic forces. Equation (22) represents the free vibration of the structural model, with a generalized mass produced by the interaction between fluid-solid domains. The coupled frequencies are given by:
\[
\omega = \sqrt{\frac{\bar{K}}{\bar{M} + \bar{M}_{\text{fluid}}}}
\]  

(23)

4. APPLICATION EXAMPLES AND RESULTS

Two application examples of the previous described procedure are presented on this item. A simple beam associated to an acoustic cavity is solved analytically and these results are compared to a finite element analysis. Figure 3 illustrates the general analysis scheme including the material and geometrical properties.

![Figure 3. General analysis scheme, material and geometrical properties](image)

4.1. Application to an open cavity with a vibrating boundary (Case 1)

In this case the pressure solution is given by Eq. (9). Substitution of this expression in Eq. (23) provides the frequency equation for a given mode shape \( \phi(y) \). Table 1 presents these results.

4.2. Application to a closed cavity in the longitudinal direction with a vibrating boundary (Case 2)

In this case the pressure solution is given by Eq. (13). Substitution of this expression in Eq. (23) provides the frequency equation for a given mode shape \( \phi(y) \). Table 2 presents these results.

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<th>( \omega_{\text{analytical}} ) (rad/s)</th>
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Table 2. Analytical and numerical solutions – Case 2

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5. CONCLUDING REMARKS

A closed analytical procedure for solution of frequencies and mode shapes of a simple beam connected to an acoustic cavity was presented. Analytical results are in agreement with analysis using the finite element method. It is important to remember that the only limitation of this procedure is the prior knowledge of the imposed deformation functions at the interface.

6. ACKNOWLEDGEMENTS

The authors are grateful for the financial support provided by CNPq scholarship.

7. REFERENCES


8. RESPONSIBILITY NOTICE

The authors are the only responsible for the printed material included in this paper.