SIMPLIFIED APPROACH FOR THE ANALYSIS OF A VISCOELASTIC PLATE IMPACT RESPONSE USING FRACTIONAL DERIVATIVE CONSTITUTIVE EQUATIONS

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Abstract. The impact of a rigid body upon an infinite isotropic plate is investigated for the case when the viscoelastic features of the plate represent themselves only in the place of contact and are governed by the standard linear solid model with fractional derivatives. Thus, the problem concerns the shock interaction of the dropping mass and the target, wherein instead of the Hertz contact law the generalized fractional-derivative standard linear solid law is employed as a law of interaction. The part of the plate beyond the contact domain is assumed to be elastic, and its behavior is described by the equations of motion which take rotary inertia and shear deformations into account. It is assumed that transient waves generate in the plate at the moment of impact, the influence of which on the contact domain is considered using the theory of discontinuities. To determine the desired values behind the transverse shear wave front, one-term ray expansions are used, as well as the equations of motion of the falling mass and the contact region. As a result, we are led to a set of two linear differential equations, the solution of which is found analytically by the Euler substitution method, what allows us to obtain the time-dependence of the contact force. Numerical analysis shows that maximum of the contact force increases tending to the maximal contact force at the fractional parameter equal to unity.

Keywords: impact, shock interaction, fractional derivative viscoelasticity, ray method

1. INTRODUCTION

Phillips and Calvit (1967) were probably the first to investigate the impact response of a viscoelastic infinitely extended plate. They used the Hertz’s contact law in its hereditary form. This problem is an immediate extension of Zener’s approach for the dynamic rigid spherical-indenter problem for the case of a thin elastic plate (Zener, 1941).

The other approach to the problem, when viscosity is included during impact, is based on replacing Hertz’s contact equation by the Maxwell equation connecting the contact force with the deformation of a viscoelastic element located between the impactor and the target. This approach was implemented by Hammel (1976) for the analysis of aircraft impact on a spherical shell.

Gonsovskii et al. (1972), when investigating the impact of a viscoelastic rod against a rigid barrier, suggested to describe the hereditary features of the rod’s material by a fractional derivative model, which was written in an equivalent form in terms of Boltzmann-Volterra relationships with a fractional exponent as a weakly singular kernel.

Rossikhin and Shitikova (2008) generalizing the approaches described in Gonsovskii et al. (1972) and Hammel (1976) proposed to use a fractional derivative Maxwell model for the analysis of the impact plate response, when its equations of motion take the rotary inertia and shear deformations into account. It is assumed that a transient wave of transverse shear is generated in the plate, and that the reflected wave has insufficient time to return to the location of the spring’s contact with the plate before the impact process is completed. To determine the desired values behind the transverse-shear wave front, one-term ray expansions are used, as well as the equations of the impactor and the contact region. The solution to this problem was found analytically by the Laplace transform method, and the time-dependence of the contact force was obtained.

In the present paper, the simplified approach developed by Rossikhin and Shitikova (2009) for the analysis of free damped vibrations of fractional derivative oscillators is implemented for the analysis of the plate response to the impact by the dropping mass, wherein instead of the Hertz contact law the generalized fractional-derivative standard linear solid law is employed as a law of shock interaction.

2. PROBLEM FORMULATION AND GOVERNING EQUATIONS

Let a rigid cylindrical body of mass \( m \) and radius \( r_0 \) with the initial velocity \( V_0 \) impact a circular Uflyand-Mindlin plate of infinite extent (this assumption is introduced due to the short duration of contact interaction in order to ignore reflected waves) with thickness \( h \). In other words, it is assumed that the impactor will bounce from the target before the reflected waves have a time to reach the place of contact. Generally speaking, the procedure to be developed in this paper allows one to consider the influence of the reflected waves on the duration of contact in the case they approach the contact place before the termination of the interaction process. However, this question will not be considered here.
To describe the process of the shock interaction of the impactor with the target with due account for the viscoelastic features, we shall use the generalized fractional-derivative standard linear solid law instead of the Hertz contact law (Fig. 1). At the moment of impact, shock waves are generated in the plate, which then propagate along the plate with the velocities of transient elastic waves.

\[
\begin{align*}
\frac{\partial Q_r}{\partial r} + \frac{1}{r} Q_r &= \rho h \dot{w}, \\
\dot{Q}_r &= K \mu h \left( \frac{\partial W}{\partial r} - B_r \right), \\
\frac{1}{r} \left( M_r - M_\phi \right) + \frac{\partial M_r}{\partial r} + Q_r &= \frac{\rho h^3}{12} B_r, \\
M_r &= D \left( \frac{\partial B_r}{\partial r} + \sigma B_r \right), \\
M_\phi &= D \left( \frac{B_r}{r} + \sigma \frac{\partial B_r}{\partial r} \right),
\end{align*}
\]

where \( r \) and \( \phi \) are the polar radius and angle, respectively, \( M_r \) and \( M_\phi \) are the bending moments, \( Q_r \) is the shear force, \( B_r \) is the angular velocity of rotation of the normal to the plate’s middle surface in the \( r \)-direction, \( W = \dot{w} \) is the velocity of plate’s deflection, \( D \) is the cylindrical rigidity, \( \rho \) is the density, \( K \) is the shear modulus, \( \mu \) is the shear coefficient, \( \sigma \) is Poisson’s ratio, and an overdot denotes the time derivative.

The equations of motion of the impactor and the contact area (Fig. 1)

\[
m(\tilde{v}_1 + \tilde{v}_2) = -F,
\]

\[
\rho h \pi r_0^2 \tilde{w}_1 = -2\pi \rho \dot{Q}_r \bigg|_{r=r_0} + F
\]

subjected to the initial conditions

\[
\tilde{w}_1 \bigg|_{t=0} = w_2 
\tilde{w}_2 \bigg|_{t=0} = w_1 
\tilde{w}_1 \bigg|_{t=0} = 0, 
\tilde{w}_2 \bigg|_{t=0} = V_0
\]

should be added to Eqs. (1) and (2), where \( w_2 \) and \( w_1 \) are the displacements of the upper and lower points of the buffer, respectively.

The contact force \( F \) is connected with the difference in displacements \( \Delta w = w_2 - w_1 \) of the buffer’s upper and lower ends by the generalized standard linear solid model (Rossikhin and Shitikova, 1997) with Riemann-Liouville derivative

\[
(1 + \tau_c D_\gamma^\gamma) F = E_0 (1 + \tau_\sigma D_\gamma^\gamma) \Delta w,
\]

\[
D_\gamma^\gamma F = \frac{d}{dt} \int_{-\infty}^{t} \frac{F(t-t')}{(1-\gamma)\Gamma(1-\gamma)} dt',
\]

where, \( \tau_c \) and \( \tau_\sigma \) are the relaxation and retardation times, respectively, \( E_0 \) is the relaxed elastic modulus, and \( \gamma \) (\( 0 < \gamma \leq 1 \)) is the order of the fractional derivative (fractional parameter). The fractional derivative representation in the form of (7) can be utilized in the cases when the transient processes can be ignored (Rossikhin and Shitikova, 2009).
Equation (6) can be rewritten as

\[ F(t) = E_0 \frac{1 + \tau_\sigma \gamma D_\gamma'}{1 + \tau_\sigma \gamma D_\gamma'} \Delta w(t) = \left[ E_x - (E_x - E_0) \frac{1}{1 + \tau_\sigma \gamma D_\gamma'} \right] \Delta w(t), \]  

(8)

where \( \tau_\sigma \gamma D_\gamma' = E_0 E_x^{-1} \), and \( E_x \) is the magnitude of the non-relaxed modulus of elasticity.

3. METHOD OF SOLUTION

Following Rossikhin and Shitikova (2008), we shall assume that during the impact process transverse forces and shear deformations predominate in the plate’s stress-deformed state in vicinity of the contact spot (the contact region of plate and buffer interaction), and use one-term ray expansions for the desired functions outside the rigid contact spot, what allows us to obtain in the vicinity of the contact region and on its boundary the following relationship between the shear force, the velocity of the quasi-transverse wave \( G^{(2)} \), and the velocity of plate’s deflection \( W \), i.e., the dynamic condition of compatibility:

\[ Q_x = -\rho G^{(2)} h W, \quad G^{(2)} = \frac{K \mu}{\rho}. \]  

(9)

Considering (5), (8), and (9), Eqs. (3) and (4) can be rewritten as

\[ \ddot{w}_1 = -B \dot{w}_1 + \Omega_\omega^2 (w_2 - w_1) - (\Omega_x^2 - \Omega_\omega^2)(1 + \tau_\sigma \gamma D_\gamma')^{-1} (w_2 - w_1), \]  

(10a)

\[ \ddot{w}_1 + \ddot{w}_2 = -\omega_2^2 (w_2 - w_1) - (\omega_2^2 - \omega_0^2)(1 + \tau_\sigma \gamma D_\gamma')^{-1} (w_2 - w_1), \]  

(10b)

where \( \Omega_x^2 = E_x M^{-1} \), \( \Omega_\omega^2 = E_0 M^{-1} \), \( M = \rho \pi T_0^2 h \), \( B = 2 T_0^{-1} G^{(2)} \), \( \omega_0^2 = E_0 m^{-1} \), and \( \omega_2^2 = E_x m^{-1} \).

The solution of Eqs. (10) we shall seek with a help of the Euler substitution

\[ w_1 = c_1 e^{\lambda t}, \quad w_2 = c_2 e^{\lambda t}, \]  

(11)

where \( c_1 \) and \( c_2 \) are arbitrary constants.

Substituting (11) in Eqs. (10) yields

\[ c_1 \left[ \lambda^2 + B \lambda + \Omega_\omega^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} \right] - c_2 \Omega_0^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} = 0, \]  

(12a)

\[ c_1 \left[ \lambda^2 - \omega_2^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} \right] + c_2 \left[ \lambda^2 + \omega_0^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} \right] = 0, \]  

(12b)

For the system (12) possesses nontrivial solution, it is a need to vanish to zero its determinant, i.e.,

\[ \begin{vmatrix} \lambda^2 + B \lambda + \Omega_\omega^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} & -\Omega_0^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} \\ \lambda^2 - \omega_2^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} & \lambda^2 + \omega_0^2 \frac{1 + (\lambda \tau_\sigma)^{\gamma'}}{1 + (\lambda \tau_\sigma)^{\gamma'}} \end{vmatrix} = 0. \]  

(13)

As a result we are led to the characteristic equation

\[ f_+ (\lambda) = \lambda^{3+\gamma} + \tau_\sigma \gamma \lambda^3 + B \lambda^{2+\gamma} + B \tau_\sigma \gamma \lambda^2 + \omega_2^2 C \lambda^{1+\gamma} + \omega_0^2 C \tau_\sigma \gamma \lambda + \omega_2^2 B \lambda^\gamma + \omega_0^2 B \tau_\sigma \gamma = 0, \]  

(14)
where $C = 1 + 2mM^{-1}$. The zero root of Eq. (13) has been discarded, since it does not appear in the exact solution. Numerical analysis of the characteristic Eq. (14) shows that it possesses two pairs of complex conjugate roots $\lambda_j$.

Substituting the proposed roots $\lambda_j$ into the set of Eqs. (12), and dropping one of its equations due to their linear dependence, we obtain

$$c_2 = \zeta(\lambda_j)c_1, \quad \zeta(\lambda_j) = -\left(1 + \frac{M}{m}\right) + \frac{M}{m}B\lambda_j^{-1} \quad (j = 1, 2, 3, 4).$$

(15)

Based on the aforesaid, the solution of Eqs. (10) takes the form

$$w_1 = c_1^{(1)}e^{-\lambda_1t} + \bar{c}_1^{(1)}e^{\lambda_1t} + c_1^{(2)}e^{-\lambda_2t} + \bar{c}_1^{(2)}e^{\lambda_2t},$$

(16a)

$$w_2 = c_1^{(1)}\zeta(\lambda_1)e^{-\lambda_1t} + \bar{c}_1^{(1)}\zeta(\lambda_1)e^{\lambda_1t} + c_1^{(2)}\zeta(\lambda_2)e^{-\lambda_2t} + \bar{c}_1^{(2)}\zeta(\lambda_2)e^{\lambda_2t},$$

(16b)

where an overbar denotes the complex conjugate of the corresponding value, and $c_1^{(1)}$, $\bar{c}_1^{(1)}$, $c_1^{(2)}$, and $\bar{c}_1^{(2)}$ are arbitrary complex constants.

If we rewrite the complex values appearing in (16) in their geometrical form, i.e.,

$$c_1^{(1)} = a_1e^{i\varphi_1}, \quad \bar{c}_1^{(1)} = a_1e^{-i\varphi_1}, \quad c_1^{(2)} = a_2e^{i\varphi_2}, \quad \bar{c}_1^{(2)} = a_2e^{-i\varphi_2},$$

$$\lambda_1 = \eta_1e^{i\psi_1}, \quad \bar{\lambda}_1 = \eta_1e^{-i\psi_1}, \quad \lambda_2 = \eta_2e^{i\psi_2}, \quad \bar{\lambda}_2 = \eta_2e^{-i\psi_2},$$

$$\zeta(\lambda_1, \lambda_2) = R_{1,2}e^{i\phi_{1,2}}, \quad \zeta(\bar{\lambda}_1, \bar{\lambda}_2) = R_{1,2}e^{-i\phi_{1,2}}, \quad \tan \Phi_{1,2} = \frac{Mm^{-1}Br_{1,2}^{-1}\sin \psi_{1,2}}{1 + Mm^{-1} + Mm^{-1}Br_{1,2}^{-1}\cos \psi_{1,2}},$$

then relationships (16) can be written in the form

$$w_1 = a_1e^{-\alpha_1t} \cos(\omega_1t + \phi_1) + a_2e^{-\alpha_2t} \cos(\omega_2t + \phi_2),$$

(17a)

$$w_2 = a_1R_1e^{-\alpha_1t} \cos(\omega_1t + \phi_1 + \Phi_1) + a_2R_2e^{-\alpha_2t} \cos(\omega_2t + \phi_2 + \Phi_2),$$

(17b)

where $a_1$, $a_2$, $\phi_1$, and $\phi_2$ are real arbitrary constants which are determined from the initial conditions (5).

Substituting (17) into the initial conditions (5), we find

$$a_1 \cos \phi_1 + a_2 \cos \phi_2 = 0,$$

(18a)

$$a_1 \eta_1 \cos(\phi_1 + \psi_1) + a_2 \eta_2 \cos(\phi_2 + \psi_2) = 0,$$

(18b)

$$a_1 \eta_1 \cos(\phi_1 + \Phi_1) + a_2 \eta_2 \cos(\phi_2 + \Phi_2) = 0,$$

(18c)

$$a_1 \eta_1 \cos(\phi_1 + \Phi_1 + \psi_1) + a_2 \eta_2 \cos(\phi_2 + \Phi_2 + \psi_2) = V_0.$$  

(18d)

Solution of Eqs. (18) gives us the unknown constants

$$\tan \phi_1 = \frac{R_{1,2}(\cos \psi_1 - \cos \Phi_1) + R_{2,2}(\cos \Phi_2 - R_{1,2}\cos \psi_2)}{R_{2,2}\sin \Phi_2 - R_{1,2}\sin \psi_2},$$

(19a)
\[
\tan \phi_2 = \frac{R_2 \eta_2 (\cos \Phi_2 - \cos \psi_2) + R_1 \eta_1 \cos \psi_1 - R_1 \eta_2 \cos \Phi_1}{R_2 \eta_1 \sin \psi_1 - R_1 \eta_2 \sin \Phi_1}, \quad (19b)
\]
\[
a_1 = -a_2 \frac{\cos \phi_2}{\cos \phi_1}, \quad (19c)
\]
\[
a_2 = V_0 \left[ \eta_1 R_1 \left[ \cos(\Phi_2 + \psi_2) - \tan \phi_2 \sin(\Phi_2 + \psi_2) \right] - \eta_2 R_2 \left[ \cos(\Phi_1 + \psi_1) - \tan \phi_1 \sin(\Phi_1 + \psi_1) \right] \right]^{-1} \frac{1}{\cos \phi_2}. \quad (19d)
\]

To determine the contact force \( F(t) \), it is a need to substitute (16) into (8). As a result, we obtain

\[
F(t) = E_0 \left[ a_1 V_1 \frac{\rho_{\alpha, \epsilon}^1}{\rho_{\epsilon}^0} e^{-\alpha t} \cos(\omega t + \phi_1 + u_1 + \chi_{\alpha}^1 - \chi_{\epsilon}^1) + a_2 V_2 \frac{\rho_{\epsilon}^2}{\rho_{\epsilon}^0} e^{-\alpha t} \cos(\omega t + \phi_2 + u_2 + \chi_{\epsilon}^2 - \chi_{\epsilon}^2) \right], \quad (20)
\]

where

\[
\rho_{\alpha, \epsilon}^{1,2} = \sqrt{1 + 2(\tau_{\alpha, \epsilon}^1)^2 \cos \gamma \psi_{1,2} + (\tau_{\alpha, \epsilon}^1)^2}, \quad V_{1,2} = \left[ 1 - \left( \frac{M}{m} + \frac{M}{m} B_{\alpha, \epsilon}^{-1} \cos \psi_{1,2} \right)^2 \right]^{\frac{1}{2}} + \left( \frac{M}{m} B_{\alpha, \epsilon}^{-1} \sin \psi_{1,2} \right)^{\frac{1}{2}},
\]

\[
\tan \chi_{\alpha, \epsilon}^{1,2} = \frac{(\tau_{\alpha, \epsilon}^1)^2 \sin \gamma \psi_{1,2}}{1 + (\tau_{\alpha, \epsilon}^1)^2 \cos \gamma \psi_{1,2}}, \quad \tan u_{1,2} = \frac{M m^{-1} B_{\alpha, \epsilon}^{-1} \sin \psi_{1,2}}{1 - \left( 1 + M m^{-1} + M m^{-1} B_{\alpha, \epsilon}^{-1} \cos \psi_{1,2} \right)}.
\]

The time-dependence of the contact force calculated by Eq. (20) is presented in Fig. 2, where the magnitudes of the fractional parameter \( \gamma \) are indicated by figures near the corresponding curves. Numerical analysis shows that maximum of the contact force increases tending to the maximal contact force at the fractional parameter equal to unity. The duration of contact of colliding bodies also increases with the increase in the fractional parameter.

![Figure 2. The time-dependence of the contact force](image)

4. DISCUSSION

The impact of a mass on a viscoelastic spring based on a rigid foundation was investigated in Chen and Lakes (1990). The process of impact was approximated as one half cycle of free decay oscillation of the one-dimensional mass-viscoelastic buffer system.

In the present paper, the viscoelastic spring is embedded into an elastic foundation (Fig. 1). If \( B \to \infty \) and \( C \to 1 \) in Eq. (14), i.e., if we suppose the foundation to be rigid, then we are led to the characteristic equation for the fractional oscillator based on the fractional derivative standard linear solid model

\[
\lambda^{2+\gamma} + \tau_{\epsilon}^{-\gamma} \lambda^2 + \omega_{\alpha, \epsilon}^2 \lambda^\gamma + \omega_{\gamma, \epsilon}^2 \tau_{\epsilon}^{-\gamma} = 0.
\]

(21)

This equation has been investigated in detail in Rossikhin and Shitikova (1997).
Equation (21) possesses two complex conjugate roots and, in contrast to the characteristic equation for the standard linear solid model with ordinary derivatives, lacks the real negative roots at any magnitude of $\tau$. In this case, the solution in the form of (11) is valid only for small viscosity ($\tau$ is small); otherwise the solution should contain one more term which is related to the relaxation-retardation processes and defines the drift of the position of equilibrium.

To study the forced vibrations of the fractional standard linear solid model oscillator, the loss tangent $\tan \delta$ of

$$\tan \delta = \frac{(E_x - E_0) \sin \psi}{E_x (\omega \tau)^\gamma + E_0 (\omega \tau)^\gamma + (E_x - E_0) \cos \psi}$$

(22)

can be used.

As it was suggested by Chen and Lakes (1990), the solution for the forced vibrations can be rewritten in terms of $\tan \delta$ in our case as well, and it is possible to investigate its influence on the contact force during impact. However, the presence of the elastic foundation severely complicates the matter, since it results in the asymmetric plot of the contact force (Fig. 2). The order of the fractional parameter also leads to the asymmetric character of the time-dependence of the contact force.

5. CONCLUSION

The impact of a rigid body upon an infinite isotropic plate is investigated for the case when the viscoelastic features of the plate represent themselves only in the place of contact and are governed by the standard linear solid model with fractional derivatives. Due to the short duration of contact interaction, the reflected waves are not taken into account, in other words, it is assumed that the impactor will bounce from the target before the reflected waves have a time to reach the place of contact.

The solution of the stated problem is found analytically by the Euler substitution method avoiding the difficulties of the Laplace transform method during the inversion from the frequency domain to the time domain.

The simplified approach proposed in this paper is fully justified in the case of small viscosity, when relaxation-retardation processes pass in a viscoelastic plate rather fast, and the drift of the position of system’s equilibrium influences weakly the damped vibrations (20) occurring around this equilibrium position. If viscosity is not small, then the given approach allows one at least to estimate qualitatively the value of the maximal contact force and the duration of contact of colliding bodies.

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7. REFERENCES


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