ASYMPTOTIC METHODS IN SINGULARLY PERTURBED MECHANICAL SYSTEMS DYNAMICS

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Abstract. One of the central problems in solving of concrete engineering tasks is a problem of modelling. The description of complex engineering objects dynamics on the language of the mathematical model is inevitably accompanied by series of additional questions of methodological and methodical nature (with using both formalized procedures and non-formalized ones of heuristic character). Therefore it is necessary the careful development of such model, that will be corresponding both to our knowledge level and to available computing possibilities.

Here the main aim is to work out the general approach applicable to different mechanical systems for solving problems of modelling. Herewith it is necessary the generalizations, that will be reflecting the most important general lines of the examined object or phenomena and – will be helping for constructing the adequate mathematical model, correct in dynamics problems. The developed approach is based on the combination of the asymptotic methods with methods of the stability theory of A.M.Lyapunov-N.G.Chetayev, and with extension of classical statements (including A.M.Lyapunov’s reduction principle, N.G.Chetayev’s stability postulate, N.N.Moiseev’s minimal models, ...) for important applications to problems of engineering practice. In this research the effective approach is established, that is considering the problem of modelling in Mechanics from united point (as s-stability problem). Besides the different mechanical systems are considered with unified view as objects of singularly perturbed class; the procedures of division of the motion of such systems on different-frequency components are worked out. And the approximate models of lower order are constructed on regular scheme as simplified systems, that are describing the motions of selected class (slow s-motion).

The methodology of stability theory in combination with asymptotic approach is very perspective, both for applied problems and in general gnosiological aspect. The developed method, based on Lyapunov’s theory, allows to consider the various mechanical systems with the unified positions; to come up to the idealization problem in mechanics (with the designing of idealized models by strict mathematical way; with the substantiation of validity of these models; with the receiving of correctness conditions in dynamics). This technique gives the regular manners for the decomposition of system. It allows to investigate the complex systems by analytical (and computer-analytical) methods in problems of analysis and synthesis.

Keywords: asymptotic models, mechanics, stability, singularity

1. INTRODUCTION

The work is devoted to the elaboration of analysis methods in the complicated systems dynamics. Generalized approach, based on the methods of Lyapunov (1956) and ideas of Chetayev (1957), considering from unified views all mechanical systems as singular ones in sense of Tikhonov (1952), Nayfeh (1973), Smith (1985), makes it possible to get the reduction principle in general qualitative analysis, to elaborate the technology of modelling in mechanics, Kuzmina (1982, 1990, 1994, 2006).

The subjects of research are the complex systems of the singularly perturbed class, generated by the examples of concrete physical substance, those lead to the non-linear, multi-dimensional dynamical problems. The difficulties in exact solving with the analytical methods bring the necessity of the model “narrowing” and of the original problems reduction to the problems of less dimensionality. For example, in theory of gyrosystems (GS) there is the well-known precessional system (PS) as shortened model of less order, Ishlinskiy (1963), Magnus (1971), Merkin (1956). This shortened model does not take into consideration the high-frequency components of motion. But the problem of precessional theory acceptability is not solved as yet. Also in stabilization and orientation systems dynamics the different shortened systems are used, Raushenbakh and Tokar (1974), as simplified models for original systems with big (small) physical parameters. But the strict validity is not examined in applied works. For this there are not the effective methods, the efficient simple algorithms. Therefore main problems are: the elaboration of systematical procedures in modelling of complex systems; the constructing of correct simplified models as comparison models by strict mathematical manners; the developing of regular methods for obtaining acceptability criteria.

First strict statements in these problems area were formulated by A.M.Lyapunov (1892), creator of classical stability theory, founder of new strong fundamental approaches both for general theory and for engineering applications. From stability investigations the ideas and results in framework of comparison method led to the reduction principle (A.M.Lyapunov, K.P.Persidskiy, I.G.Malkin) and to the comparison principle (V.M.Matrosov, R.Bellman,...). There is close connection between these problems of singular systems and the problems of stability theory (I.M.Gradstein,
N.G.Chetayev); between problems of stability theory and problems of modelling in Mechanics (N.G.Chetayev, L.K.Kuzmina).

Here above formulated problems are considered, ones are solved by methods of Stability Theory. General approach, that was founded by A.M.Lyapunov, used by N.G.Chetayev to Mechanics problems, added by P.A.Kuzmin to statement for stability with parametric perturbations, is worked out for systems of singular class.

2. BASIC PROPOSITIONS

We shall consider the mechanical systems, taking the Lagrange’s equations (or their generalized form) as initial mathematical model

\[
\frac{d}{dt} \frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = Q, \quad \frac{dq}{dt} = \dot{q} \quad (1)
\]

\(q\) is \(k\)-dimensional vector of generalized coordinates; \(k\) is the number of freedom degrees; \(T\) is kinetic energy of system. Equations (1) is full model (FM).

In case of gyroscopic systems (GS), that are modeled by mechanical systems with the fast rotors, assuming the eigen kinetic moments of gyroscopes are big, researchers introduce big parameter \(H\) and go from FM to simplified model (SM) (precessional model). This SM has \((k/2)\)-degrees of freedom. Here we have the transition to model of lower order, and it gives non-regular problems.

GS:  \(\text{FM}(k) \to \text{SM}(k/2)\)  

Also for the stabilization systems, modeled as electromechanical systems (EMS) with \(n\)- mechanical and \(u\)-electrical numbers of freedom degrees, in the analysis of the stabilization state, assuming the follow-up systems are quick-operating, they introduce small parameter \(\mu\) (corresponding to small constants of time for electrical circuits) and go to simplified model SM. One does not take into consideration the fast electrical processes, and it has the less number of freedom degrees. On the other hand, for the investigations of dynamical processes in follow systems they introduce another SM (with the idealized inertialess mechanical subsystem). Here also we have non-regular problems.

EMS: \(\text{FM}(n+u) \mu \to \text{SM}(n+(u/2))\)  

In robotic-systems (RS) dynamics the researchers go to simplified models of lower order (for example, neglecting some real properties, connected with the non-absolute rigidity of some links or with small inertia of some elements, etc.).

RS: \(\text{FM}(k) \mu \to \text{SM}(k_1) (k_1< k)\)  

\(H_1, \mu_1\) are big, small parameters; \(k_1, k_2\) are the numbers of freedom degrees of SM.

But the legitimacy of SM and their admissibility domains are non-discussed; the conditions of acceptability are non-determined; these models are non-obtained by strict mathematical way.

Returning to the main problems, we note a general peculiarities of mechanical systems, modeled as Eq. (1): direct introducing of small (big) parameter in initial mathematical model is non-useful (it is necessary the special transformation of variables); mechanical systems are singular systems with \(\mu\) parameter in different powers; the motions in systems split up into three (or more) groups (if \(\mu \to 0\)); as rule, the idealized (simplified) models are not limit models; the mechanical systems are non-Tikhonov’s systems (unperturbed systems are on boundary of stability domain; eigenvalues of corresponding matrices are zero or imaginary ones).

Therefore the direct use of known results of singular perturbations theory are non-suitable. New methods and approach are necessary for mechanical singular problems, Voronov (1985).

3. METHODOLOGICAL ASPECTS

The general approach, founded by A.M.Lyapunov, combining the methods of perturbations theory and of stability theory, is developed. Such approach permits to detour the above indicated features.

Let the original mathematical model is Eq. (1) (full model); \(x=(q, \dot{q})\) is vector of the system state variables. The problems: how construct the correct simplified models by strict method; how determine the acceptability conditions of SM in qualitative analysis; how obtain the errors estimations. The stages:

a. First of all we must introduce the small parameter \(\mu > 0\), and converting to new variables \(y (x \to y, y = L(\mu, x))\) we shall lead the system (1) to the form
\[ M(\mu) \frac{dy}{dt} = Y(t, \mu, y), \quad M(\mu) = \left[ M_{ij}(\mu) \right], \quad M_{ij}(\mu) = \mu^i \mu^j I, \quad M_{ij}(\mu) = 0 \quad (i \neq j) \]  

(2)

Here \( y \) is a \( 2k \)-dimensional vector, \( \alpha_i = 0, 1, 2, \ldots, r; \quad r > 0 \) is constant number; \( I \) is identity matrix.

Such manner allows to obtain the effective regular technique for the constructing of simplified models (SM). For this we shall introduce as approximate system for Eq. (2) the shortened one (SS)

\[ M_0(\mu) \frac{dy}{dt} = Y_0(t, \mu, y) \]  

(3)

System (3) is obtained from Eq. (2) with the keeping of members, no higher than \( \mu^s \)-order (0 \( \leq s < r \)) (s-approximation on \( \mu \)).

System (3) is the \( N_s \)-order comparison system for Eq. (2) \( (N_s < N; N=2k) \). Returning to the state variables \( x \), we shall obtain the SM, (designated Eq. (4) without writing): \( y \to x \) and (3) \( \to (4) \). SM has \( k_s \) of freedom degrees \( (k_s = N_s/2; k_s < k) \); in general case \( k_s \) is non-integer number.

We shall call Eq. (4) the simplified model of \( s \)-level on \( \mu \)-parameter (SMs). In view point of mechanics the model (4) is some idealized model. Moreover, we can receive the hierarchical sequence of simplified models on \( \mu \)-parameter assigning \( s \) the values to 0, 1, 2, \ldots: \( SS_0, SS_1, \ldots \). Such manner allows to obtain the effective regular technique for the constructing of simplified models (SM). For this we shall introduce in consideration all shortened subsystems, accordingly to all variables domains; with the errors determination. The regular methods, based on Lyapunov’s methods (first, second), give the necessary conditions, under which the solution of (5)

\[ \Delta M(\mu) \frac{db}{dt} = B(t, \mu, b) \]  

(5)

Here \( y = y(t, \mu) \) is the full system (2) solution with initial conditions \( y_0 = y(t_0, \mu) \); \( y' = y'(t, \mu) \) is the shortened system (3) solution with corresponding data \( y_0' = y'(t_0, \mu) \). Using stability methods, analyzing system (5), we shall determine the necessary conditions, under which the solution of (5) \( b = b(t, \mu) \) will have the required properties (with small enough \( \mu \), \( \mu \leq \mu^* \)).

Here we must solve the singular problems: stability problem (under which conditions the stability property for system (3) (for SM will entails this property for system (2)) (for FM); solutions proximity problem (under which conditions the corresponding solutions of shortened system and full system will be close on infinite time interval); note, this problem is reduced to the problem of set stability; other singular problems (on speedness; of optimal parameters; etc.).

The next stage is the substantiation of the constructed model acceptability. We must determine the correctness conditions of shortened models, of their qualitative equivalence with the full model. In conformity with the stability theory ideology we introduce the differential equations for the deviations \( b = y-y' \), Chetayev (1957).

4. APPLICATIONS TO MECHANICS
Elaborated technique enables to reveal a general rule for mechanical systems: the differential equations can be represented in form (2); the variables can be separated on three groups; the idealized models can be constructed as $SM_0$ and $SM_1$ (limit model and linearized model, as $s$-approximations on $\mu, s = 0$ and $s = 1$). The correctness conditions of these idealizations are determined by developed methods.

As illustration we consider these general propositions on the concrete mechanical system of stabilization, modeled as EMS. Let the mathematical model is the Lagrange-Maxwell’s equations, written on the form (1)

$$\frac{d}{dt}a_qq_M + b_Mq_M + gq_M = Q_M + Q_{ME} + Q_s,$$

$$\frac{d}{dt}a_qq_M + b_qq_R = Q_E + Q_{EM} + Q_e + dM = q_M \tag{6}$$

$q_M, q_e$ are $n-, u$-dimensional vectors of generalized coordinates (of Lagrange, Maxwell).

System (6) is the $(2n+u)$-order one (full model, FM). For the simplification of FM we shall use our scheme $(a, b, c)$.

We shall consider the concrete cases, with concrete physical assumptions.

* EMS with the fast gyroscopes ($q = q^* H, H > 0$ is big parameter; $H = 1/\mu, \mu > 0$ is small parameter).
* EMS with the quick operating followers ($a_q = a_q \mu_1, A_M = A'M\mu, B_q = B\mu_\mu; \mu > 0$ is small parameter).
* The systems with non-rigid elements (FM is the Lagrange’s equations; the small parameter is $\nu = 1/H$, $H > 0$ is big parameter; $c = c^* H^2$, $c$ is the matrix of the potential energy of elasticity forces).
* …………………………………

In all considered cases using developed approach the variables transformation is constructed; $SM_0, SM_1$ are obtained and substantiated; new interesting results are received.

5. CONCRETE EXAMPLE

As an example of such mechanical system we shall consider the systems of gyrostabilization, modelling ones as mechanical (electromechanical) systems with controlling gyroscopic elements. Here there is a critical case of zero roots. We shall solve a stability problem of the steady motion for such system, supposing that the elements of the system are not absolutely rigid (we neglect the mass of elastic elements); and supposing that follow-up systems are idealized ($\mu = 0$). Differential equations of perturbed motion we shall accept in a form of Lagrange’s equations, Merkin (1956), Kuzmina (1990, 2006),

$$\frac{d}{dt}a_qq_M + (b + g)q_M + cM = Q^*_M \tag{7}$$

$$\frac{dq_M}{dt} = q_M$$

Here $q_M = [q_1, q_2, q_3, q_4]^T$ is $n$ - dimensional vector of mechanical generalized coordinates, where $q_1$ is $l$-dimensional vector of the gyroscopes precessions angles; $q_2$ is $(m$-$l$)-dimensional vector of angles deviations of own rotations of gyroscopes from their values in steady motion; $q_3$ is $(s$-$m$)-dimensional vector of stabilization angles, $s = m + 1$; $q_4$ is $(n$-$s$)-dimensional vector of elastic elements deformations; $a, b, g$ are square $n \times n$ -matrices of forms of the system kinetic energy, dissipative function of friction forces, gyroscopic coefficients accordingly; $c = [c_{ij}], b = [b_{ij}]$ $(i, j = 1, ..., 4)$, $c_{ij}$ and $b_{ij}$ are submatrices of an appropriate sizes; $b_{44}$ is square $(n$-$s)$-$(n$-$s$) -matrix of dissipative function of internal friction forces in material of elastic bodies; $c_{44}$ is square $(n$-$s)$-$(n$-$s$) -matrix, corresponding to potential energy of elasticity forces.

We assume that all functions in Eq. (7) are holomorphic (on the totality of their variables) in certain area.

For solving of this problem (and choosing of a reduced model) we shall lead Eq. (7) to a form (1) with singular perturbations. For this, first, we must introduce in Eq. (7) a small parameter, using physical considerations. We suppose, that the elements of the considered systems are of a sufficiently high rigidity and according to that $c_{44} = c_{44}/\mu^2$, $b_{44} = b_{44}/\mu$, where $\mu > 0$ is a small parameter. Now, using the special transformation of variables

$$z = [a_1, a_2] [q_1] [q_2] [q_3] [q_4],$$

$$\kappa_1 = [q_1, a_2, a_1] [q_1], \quad \kappa_2 = a_2 q_4, \quad q_4 = q_4, \quad (j = 1, 4)$$

where $a_1, b, g$ $(i = 1, ..., 4)$ are submatrices of matrices $a, b, g$ correspondingly, we shall lead Eq. (7) to the singularly perturbed form. This transformation is the non-linear, non-singular under condition that $|b_{ij} + g_{ij}|^{1/2} \neq 0$, evenly regular, not changing the statement of the stability problem. System (7) in new variables has a form (2)
\[
\begin{align*}
\frac{dz}{dt} &= Z(t, \mu, z, x) \\
M(\mu) \frac{dx}{dt} &= P(\mu) + X(t, \mu, z, x)
\end{align*}
\] (8)

where \( x = [x_1, x_2, x_3]^T \), \( x_1 = [x_2, \ldots, x_n]^T \), \( x_2 = x_3 = q_4; \alpha_1 = 0, \alpha_2 = 2, \alpha_3 = 0; \) \( P_2(\mu) = \mu P_2(\mu) \) \((i = 1, 2)\).

The characteristic equation has \( m \) zero-roots. Other roots can be found from the equation \( d(\lambda, \mu) = 0 \). We assume the shortened system of 0-level (degenerated system) as an approximate one for a system (8), marking it (8') without writing. In old variables it is the system

\[
\frac{d}{dt} a^* \dot{q} + (b^* + g^*) \dot{q} + c^* q = Q^*, \quad \frac{d}{dt} q = \dot{q}
\] (9)

where \( q = [q_1, q_2, q_3]^T \) is \( s \)-dimensional vector of generalized coordinates, describing the state of an absolutely rigid system; \( a^*, b^*, c^*, g^* \) are \( s \times s \)-matrices of absolutely rigid system.

The equation (9) describes a motion of an idealized model of mechanical system. This model corresponds to an approximate system (8') of 0-level. We shall call it a “limit model”. A problem: in what conditions a transition from the initial model (7) to its idealized model (to absolutely rigid system) is possible? Using methods of stability theory combined with the singular perturbations methods and introducing the differential equations for deviations that respond to non-critical (basic) variables \( x \), we can find out the acceptability conditions for transition validity from system (8) to the system (8') in concrete dynamical problems. After returning to old variables, taking into account the properties of the considered mechanical system, we receive the corresponding statements.

5.1. Stability problem

When the stability property for reduced model (9) will be ensuring same property for original (full) model (7)?

**Theorem 1.** If \( \left| b_{ij}^0 + g_{ij}^0 \right|_{i=1,2} \neq 0, \left| c_{ij}^0 \right| \neq 0 \) and all roots (except \( m \) zero roots) of characteristic equation of reduced system (9) have negative real parts, then with sufficiently small values of \( \mu \) (sufficiently high rigidity of the system elements) the zero solution stability of the full system will be succeeding from the zero solution stability of reduced system (9). And reduced system (9) has integral

\[
\begin{bmatrix}
\alpha_1^* \\
\alpha_2^* \\
\end{bmatrix} \dot{q} + \begin{bmatrix}
b_{11}^0 + g_{11}^0 \\
b_{21}^0 + g_{21}^0 \\
\end{bmatrix} q + \varphi(q, \dot{q}) = B
\]

and full system (7) has integral of Lyapunov:

\[
\begin{bmatrix}
\alpha_i^* \\
\alpha_i^* \\
\end{bmatrix} \dot{q}_M + \begin{bmatrix}
b_{11}^0 + g_{11}^0 \\
b_{21}^0 + g_{21}^0 \\
\end{bmatrix} q_M + F(q_M, \dot{q}_M) = A
\]

5.2. Proximity problem

When the solutions of original system (7) and of reduced model (9) are close on infinite time interval?

Let \( q_i = q_i(t, \mu), \dot{q}_i = \dot{q}_i(t, \mu), (i = 1 \ldots 4) \) be the solution of system (7) with the initial conditions \( q_{i_0} = q_{i_0}(t_0, \mu), \dot{q}_{i_0} = \dot{q}_{i_0}(t_0, \mu) \); we shall designate \( \dot{q}_i^* = \dot{q}_i^*(t), \dot{q}_i^* = \dot{q}_i^*(t), (i = 1 \ldots 4) \) as the solution of system (9), defined by the initial conditions \( q_{i_0} = q_i^*(t_0), \dot{q}_{i_0} = \dot{q}_i^*(t_0) \) \((i = 1, 2, 3)\), where \( \dot{q}_i^* = 0, \dot{q}_i^* = 0 \).

Making use of stability theory methods we can prove the following statement:

**Theorem 2.** If the characteristic equation for system (9) has all roots in the left half-plane except \( m \) zero roots for \( d(0, 0) = 0 \), then under sufficiently big stiffness of the system elements (i.e. \( \mu \) is sufficiently small) there exists such a \( \mu_* \)-value for \( \xi > 0, \eta > 0, \gamma > 0 \) given in advance (no matter how small \( \xi \) and \( \gamma \) are), that in a perturbed motion:
\[
\begin{align*}
\|q_i - q^*_i\| &< \xi, \quad \|\dot{q}_i - \dot{q}^*_i\| < \xi \quad (i=1,\ldots,4) \quad \text{when } 0 < \mu < \mu^*, \quad \forall t \geq t_0 + \gamma, \quad \text{if} \quad q_{j0} = q_{j0}^*, \quad \dot{q}_{j0} = \dot{q}_{j0}^*, \quad (j=1,2,3) \\
\|\hat{q}_0\| &< \eta, \quad \|\dot{\hat{q}}_0\| < \eta.
\end{align*}
\]

It should be pointed out that while demonstrating and using variables \(z, x\) we introduce deviations \(a = z - z^*, b = x - x^*\) and consider a differential equation for \(b\). The analysis of these equations as well as the integral structure enable to derive the statement of Theorem.

These results, complementing already known, justify for the systems, considered here, admissibility of reduced limit model (as asymptotic model of 0-level) and determine the conditions, under which the considered transition is correct (in a meaning, adopted here).

**Remark.** According to this we can introduce other reduced model (as designing-basic model) for (7). This is asymptotic model of 1-level (s-approximation, \(s=1\)), that has \((s+(n-s)/2)\) of freedom degrees (if in (8) take into consideration members containing \(\mu\) in power not more than 1). This model is new one (it is very interesting result).

System (8) belongs to the special critical case, when all eigenvalues of matrix \(P_{22}\), corresponding to the fast \(x_2\), are zero.

### 6. CONCLUSION

The introducing of hierarchy in the sequence of the constructed approximate models is allowing to realize the decomposition of the original model, with transition to different simplified models, including limit model. The general methods for obtaining of sufficient conditions, under which the transition to the constructed simplified models will be correct (in dynamics problems: stability, operativeness, optimality) are developed in this approach. Also it is worked out the regular manners for estimation of permissible physical parameters values. Different mechanical systems, that are of special interests for the applications (mechanical systems with non-rigid elements, systems with big friction, electromechanical systems with delay, systems with quick rotors, non-holonomic systems, ...) are considered as illustration; asymptotic models are constructed; acceptability conditions are obtained (both for known traditional models and for new ones), with revealing minimal models.

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### 7. REFERENCES


### 8. RESPONSIBILITY NOTICE

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