INSTABILITY OF FLOW IN A CHANNEL WITH DISTRIBUTED HEATING

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Abstract. The linear stability of channel flow between two horizontal parallel walls in the presence of distributed heating has been investigated. The case of periodic heating applied at the bottom wall has been considered in details. This heating results in the creation of zones of fluid with alternatively increased and decreased temperature. The mean flow and the linear stability equations have been solved using spectral methods. Two types of instability, i.e., vortex instability and traveling wave instability, have been examined. For the traveling wave instability, two and three dimensional oblique waves have been considered. It has been found that from among various possible forms of disturbances the streamwise vortices appear at the lowest value of the Rayleigh number if the flow Reynolds number is sufficiently small, and two-dimensional traveling-waves appear first if the flow Reynolds number is sufficiently large. The conditions when the two-dimensional waves dominate are similar to those found in the case of an isothermal flow.

Keywords: natural convection, buoyancy effects, flow control

1. INTRODUCTION

A horizontal layer of fluid either heated from below or cooled from the above represents a classical problem in fluid mechanics. It is known that a secondary flow appears when the temperature difference reaches a critical value. When a fully developed transverse flow is added, the secondary flow has the form of longitudinal vortex rolls (Mori and Uchida, 1966). It is known that below the critical point heat is transported between the walls by conduction and the temperature changes linearly across the channel. Above the critical point the heat transfer rate is increased by the thermal instability and the temperature field is strongly influenced by the presence of vortex rolls (Ostrach and Kamotani, 1975).

As might be expected, when moving from the case of an initially stagnant fluid to that of a moving fluid, an experimental analysis can most readily be used to investigate the physics of the flow. Mori and Uchida (1966) did just this for the instance of a fluid heated from below. They quantified both the critical parameters for the formation of rolls, oriented with their axis in the direction of the flow, and the associated increase in the transfer of heat between the upper and lower walls of the channel due to the improved mixing provided by these rolls. Pellow and Southwell (1940) took an analytical approach to the problem of an unstably stratified flow in a horizontal channel. Their approximation of the increase in heat transfer was shown to match well with the experimental data available, as well as the critical flow parameters derived for the formation of longitudinal rolls. Ostrach and Kamotani (1975) and Silveston (1958) undertook further experiments in which the flow was directly observed during the formation of the rolls.

As an extension to the classical case of a flowing fluid heated from below, Akiyama et al. (1971) experimentally investigated this arrangement with an additional thermal gradient in the streamwise direction and Nakayama et al. (1970) provided a numerical simulation. The application of this work to heat exchanger optimization is apparent in the presentation of the result in both cases. A good summary of the behavior of thermally stratified fluid in the presence of a viscous shear flow is given by Gage and Reid (1968). It includes a discussion of the form of initial instability, in particular whether rolls or waves will form, as well as treating both the case of uniform heating from below and the case of uniform heating from above. The three-dimensional character of the instabilities is examined spatially with regard to the orientation of disturbance structures in the channel.

This paper deals with the linear stability analysis of Boussinesq fluid confined in a channel with bottom wall subject to spatially distributed heating and a fully developed transverse flow. A complete formulation of the problem is presented and some results are shown.

2. FLOW PROBLEM FORMULATION

This section deals with description of the modification of the Poiseuille flow due to the presence of distributed heating. Consider flow confined in a channel bounded by walls at \( y = \pm 1 \) and extending to \( \pm \infty \) in the \( x \) and \( z \)-directions as shown in Fig.1. The reference flow is Poiseuille flow directed along the positive \( x \)-axis and driven by a pressure gradient. The fluid is incompressible and Newtonian. This flow is modified by distributed heating applied at the bottom wall resulting in the temperatures of the walls in the form
\[ \theta_L(x, z) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} L_n^{(m,n)} e^{i(\alpha x + \beta z)}, \quad \theta_L(x, z) = 0 \]

where \( \alpha = \frac{2\pi}{\lambda_x} \), \( \beta = \frac{2\pi}{\lambda_z} \) and \( \lambda_x \) and \( \lambda_z \) denote wavelengths in the streamwise and spanwise directions, respectively, and the ‘hat’ symbol denotes Fourier coefficient. \( \theta \) denotes the relative temperature, i.e., \( \theta = T - T_{\text{ref}} \). The dimensional field equations for a Boussinesq fluid are scaled using two sets of scales, i.e., one to characterize the Poiseuille flow and the other to characterize the convective structures resulting from the spatially distributed heating.

\[
\begin{align*}
U(x, z) &= 0 \quad (1a,b) \\
\theta &= \frac{T - T_{\text{ref}}}{T_{\text{ref}}} \\
\end{align*}
\]

where \( \alpha = 2\pi / \lambda_x \), \( \beta = 2\pi / \lambda_z \) and \( \lambda_x \) and \( \lambda_z \) denote wavelengths in the streamwise and spanwise directions, respectively, and the ‘hat’ symbol denotes Fourier coefficient. \( \theta \) denotes the relative temperature, i.e., \( \theta = T - T_{\text{ref}} \). The dimensional field equations for a Boussinesq fluid are scaled using two sets of scales, i.e., one to characterize the Poiseuille flow and the other to characterize the convective structures resulting from the spatially distributed heating.

The velocity and pressure fields associated with the Poiseuille flow have the form

\[ [U(y), 0, 0] = [1 - y^2, 0, 0], \quad P(x, y, z, t) = -2x / \text{Re} \quad (2) \]

where Reynolds number, \( \text{Re} \) is defined based on the maximum velocity of the Poiseuille flow and the channel half-height. The flow quantities are assumed to have the following form

\[
\begin{align*}
u_2(x, y, z) &= \text{Re} U(y) + u_1(x, y, z), \\
v_2(x, y, z) &= v_1(x, y, z), \\
\theta_2(x, y, z) &= \text{Pr}^{-1} \theta_0(x, y, z) + \theta_1(x, y, z), \\
P_2(x, y, z) &= \text{Re}^2 P_0(x, y, z) + \text{Pr} P_1(x, y, z) \\
\end{align*}
\]

where the subscript 1 denotes flow modifications due to the distributed heating and \( \theta_0 \) is the conduction temperature field defined as

\[ \theta_0(x, y, z) = \theta_{0,\text{con}}(y) + \theta_{0,\text{num}}(x, y, z) \quad (4) \]

where \( \theta_{0,\text{con}} \) stands for the part of temperature field generated by the mean temperature differences between the channel walls and \( \theta_{0,\text{num}} \) stands for the spatially non-uniform part described by the steady state heat conduction equation. The non-uniform part of temperature can be expressed as,

\[ \theta_{0,\text{num}}(x, y, z) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \left[ -\hat{L}_n^{(m,n)} \frac{\sinh(k_n y)}{2} + \hat{L}_n^{(m,n)} \frac{\cosh(k_n y)}{2} \right] e^{i(k_n x + m\beta z)} \quad (5) \]

where

\[ k_n^2 = n^2 \alpha^2 + m^2 \beta^2. \]

The flow is assumed to be periodic in the streamwise and spanwise directions leading to flow quantities in the form of Fourier expansion, e.g.
\[ \Phi(x, y, z, t) = \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} \Phi_{nm}(y, t) e^{(i\alpha x + \beta m z)} \]  

where \( \Phi \) stands for any flow quantity (e.g., \( u, v, w, \theta \) etc.). The final form of the steady governing equations expressed in terms of the wall-normal vorticity \( \eta = \frac{\partial U}{\partial z} - \frac{\partial W}{\partial x} \), wall-normal velocity and temperature have the form:

\[ (D_{n,m} - i \alpha Re U) \hat{\eta}_{n,m} + (n \alpha + \beta) Re \frac{dU}{dy} \hat{\eta}_{n,m} + \hat{N}_{n,m} = 0, \]  

\[ \left[ D^2_{n,m} + i \alpha Re \left( \frac{-U D_{n,m}}{dy^2} + \frac{d^2 U}{dy^2} \right) \right] \hat{\eta}_{n,m} - k^2_{n,m} Re \left[ Pr^{-1} \hat{\eta}_{n,m} + \hat{\Theta}_{n,m} \right] + \hat{N}_{n,m} = 0, \]  

\[ Pr^{-1} D_{n,m} \hat{\Theta}_{n,m} - i \alpha Re U \left[ Pr^{-1} \hat{\Theta}_{n,m} + \hat{\Theta}_{n,m} \right] - \Pr^{-1} \hat{\Theta}_{n,m} - \hat{N}_{n,m} = 0, \]  

where

\[ D_{n,m} = d^2 / dy^2 - k^2_{n,m}, \quad D = d / dy, \]

and the symbol \( N \) stands for the corresponding nonlinear terms whose explicit forms are not shown. Here, \( Pr = \nu / \kappa \) and \( Rayleigh \) number, \( Ra = g \gamma d^3 \Delta T / \kappa \nu \), where, \( \nu \) is the kinematic viscosity, \( \kappa \) is the thermal diffusivity, \( g \) stands for acceleration due to gravity, \( \gamma \) is the volumetric coefficient of expansion, \( d \) is the channel half-height, and \( \Delta T \) denotes temperature difference. Equations (7-8) are subject to the no-slip and no-penetration conditions at both walls. The boundary condition for temperature is simply \( \hat{\Theta}_{n,m}(0) = 0 \) at both walls. The field equations (7-9) subject to the above boundary conditions are solved using spectral collocation method in the \( y \)-direction.

3. FORMULATION OF THE LINEAR STABILITY PROBLEM

The analysis begins with the governing equations in the form of vorticity transport, energy and continuity equations. Unsteady three-dimensional disturbances are super-imposed on the mean part. The forms of the disturbance field can be written as, according to Floryan (1997),

\[ \bar{\varphi}_3(x, y, z, t) = \sum_{l=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} \left[ g_n^{(l)}(y) g_v^{(l)}(y) g_w^{(l)}(y) \right] e^{(i\delta_j + i\alpha x + \beta j z - \omega t)}, \]  

\[ \theta_3(x, y, z, t) = \sum_{l=-\infty}^{\infty} \left[ g_0^{(l)}(y) \right] e^{(i\delta_j + i\alpha x + \beta j z - \omega t)}. \]

Substitution of (10) into the linearized field equations and separation of the Fourier components results, after rather lengthy algebra, in a system of linear ordinary differential equations in the form

\[ T^{(l)} \zeta^{(l)} + \beta (Re D U) g_v^{(l)} = \sum_{n=-\infty}^{\infty} \left[ E^{(l-n)}\zeta^{(l-n)} + E_v^{(l-n)} g_v^{(l-n)} \right], \]

\[ S^{(l)} g_v^{(l)} - Ra K^2 \theta_0^{(l)} = - \sum_{n=-\infty}^{\infty} \left[ H^{(l-n)}\zeta^{(l-n)} + H_v^{(l-n)} g_v^{(l-n)} \right], \]

\[ P^{(l)} \theta_0^{(l)} = Pr \sum_{n=-\infty}^{\infty} \left[ J^{(l-n)}\zeta^{(l-n)} + J_v^{(l-n)} g_v^{(l-n)} + J_0^{(l-n)} \theta_0^{(l-n)} \right], \]

\[ \nu \epsilon \zeta^{(l)} + D g_v^{(l)} + \epsilon \beta g_w^{(l)} = 0 \]

where

\[ T^{(l)} = D^2 - k_l^2 - i(t_l (Re U - C) - \alpha), \quad k_l^2 = t_l^2 + \beta^2, \quad t_l = \delta + i\alpha \],

\[ S^{(l)} = \left( D^2 - k_l^2 \right) - i(t_l (Re U - C) - \alpha) \left( D^2 - k_l^2 \right) + \nu (Re D)^2 U, \]
\[
P^{(l)} = D^2 - k^2 - iPr(t_l(ReU - C) - \sigma),
\]
\[
\gamma^{(l)} = t_l\gamma_u^{(l)} - \beta g_u^{(l)},
\]
\[
E_v^{(l-n)} = it_l f_u^{(n)} + k_{l-n}^2 \left( \beta^2 + t_{l-n} t_l \right) f_v^{(n)} D,
\]
\[
E_{y}^{(l-n)} = -\beta D f_y^{(n)} + ina \beta k_{l-n} f_v^{(n)} D^2,
\]
\[
H_v^{(l-n)} = nax^2 \left( t_{l-n} f_u^{(n)} D + \left( t_l + t_{l-n} \right) f_y^{(n)} - ik_2^2 f_v^{(n)} D^2 \right),
\]
\[
H_{y}^{(l-n)} = k_{l-n}^2 \left( \beta^2 - t_l t_{l-n} \right) f_u^{(n)} D + k_2^2 \beta^2 + \left( n\alpha t_l - k_2^2 \right) f_v^{(n)} D^2 + \left( n\alpha t_{l-n} - k_2^2 \right) f_y^{(n)} D^2 + ik_2^2 t_{l-2n} f_u^{(n)} + it_l D^2 f_u^{(n)},
\]
\[
J_{l}^{(l-n)} = -lna \beta k_{l-n}^2 f_y^{(n)},
\]
\[
J_{y}^{(l-n)} = -nax^2 t_{l-n} f_y^{(n)} D + D f_y^{(n)},
\]
\[
J_{l}^{(l-n)} = it_{l-n} f_u^{(n)} + f_v^{(n)} D,
\]
\[
f_u^{(n)} = \tilde{f}_u^{(n,0)},
\]
\[
f_v^{(n)} = \tilde{f}_v^{(n,0)},
\]
\[
f_y^{(n)} = Pr \tilde{f}_y^{(n,0)} + \tilde{f}_y^{(n,0)}.
\]

Effects of the distributed wall-heating are contained in the terms on the right hand side of (11-13). In their absence, all modes from the Fourier series (10) decouple and equation (11 -12) describe the classical three-dimensional instability of the isothermal Poiseuille flow. Equations (11-14) together with the homogeneous boundary conditions have nontrivial solution only for certain combinations of parameters Re, Pr, Ra, \(\delta\), \(\alpha\), \(\beta\) and \(\sigma\). For numerical calculations, the problem is posed as an eigenvalue problem for \(\sigma\). The above equations are discretized with spectral accuracy using Collocation method. For the purpose of eigenvalue tracking a classical Newton-Raphson search procedure is used. A reasonable guess for the unknown eigenvalue is essential for the convergence of the search routine.

![Figure 2: Neutral stability curves for channel with (a) uniformly heated bottom wall (left), (b) periodically heated bottom wall (right).](image)

### 4. RESULTS AND DISCUSSION

The analysis begins with the plane isothermal Poiseuille flow. It is known that the plane Poiseuille flow becomes linearly unstable at \(Re=5772.2\) and the critical disturbances has the form of a two-dimensional wave traveling in the streamwise direction. Such waves are typically referred to as the Tollmien-Schlichting (TS) waves. A uniform heating is applied at the bottom wall and change in the flow response has been investigated. Two types of instability, i.e., vortex and traveling-wave, have been considered. For the three-dimensional waves the degree of obliqueness has been defined as \(\Phi = \tan^{-1}(\beta/\delta)\), where \(\Phi = 0^\circ\) corresponds to the two-dimensional wave and \(\Phi = 90^\circ\) corresponds to the vortex.
Figure 2(a) shows the neutral curves of various three-dimensional waves. Similar results were obtained by Gage and Reid (1968).

As a next step, effects of spatially distributed heating have been explored. The results are illustrated for the simplest possible heating in the form of a single Fourier mode, i.e., \( \theta_L = \cos(\alpha x) \). In defining the Rayleigh number for this type of heating, the temperature difference \( \Delta T \) corresponds to the amplitude of temperature variations along the lower wall. Neutral stability curves similar to those computed for the uniform heating are displayed in Fig. 2(b) for the two dimensional waves, for different oblique waves and for the vortex. It is evident from this figure that disturbances in the form of vortices are the most unstable and the two dimensional waves are the most stable. It is also evident that for \( \text{Re}=5772.2 \) the heating has no effect on onset of instability, because at this Reynolds number TS waves starts to appear as in the isothermal flow (see Fig. 2a). When \( \text{Re}>5772.2 \), two dimensional waves become most unstable. Figure 3 illustrates the effect of obliqueness of the waves with Reynolds numbers. This figure gives a clear indication that for all Reynolds numbers considered in this study, disturbances in the form of vortices (angle of obliqueness=90°) have the lowest critical Rayleigh number.

![Figure 3: Effect of obliqueness of the waves on the critical Rayleigh number at selected values of the Reynolds number.](image1)

![Figure 4: Effect of the heating wave number on the critical conditions for the onset of vortices.](image2)

![Figure 5: Stability diagram for rolls at \( \text{Re}=0, Ra=3980 \).](image3)

![Figure 6: Neutral stability diagram for rolls at selected Ra for \( \text{Re}=0 \).](image4)

Figure 4 illustrates the effect heating wave number for different values of Reynolds number on the critical conditions for the onset of vortices. It can be seen that an increase of \( \alpha \) increases critical value of Ra. This is because of
the fact that in this case the heating effect is prominent near the bottom wall and the rest of the channel is almost isothermal.

As a final step the stability characteristics of a stagnant fluid (Re=0) subject to a periodic heating has been investigated. Figure 5 shows the stability diagram for the rolls at Ra=3980. Figure 6 shows neutral stability curves for such rolls at selected values of the Rayleigh number. It is noted that flow is unstable inside the closed curves and stable outside the closed curves. An increase of Ra results in an increase of “window of opportunity” in the $\alpha$-direction where rolls can be observed. Heating with a either a too small wave number or a too large wave number cannot lead to the onset of the rolls at these values of Ra.

5. CONCLUSIONS

Stability of channel flow modified by distributed heating applied at the lower wall has been analyzed. Here, the simplest form of such heating, i.e. periodic heating, has been considered. The analysis consists of two steps, i.e., a steady flow modified by distributed heating is determined followed by the linear stability analysis. The stability analysis properly accounts for the spatial modulation of the flow generated by the heating. Results show that disturbances in the form of streamwise vortices are the most unstable and two dimensional waves are the most stable for Re=5772 while two-dimensional waves becomes most unstable for Re=5772. The critical Rayleigh number for the onset of vortices increases approximately linearly with the increase of the flow Reynolds number. The conditions under which two-dimensional waves begin to dominate the instability process are almost identical with those found in the case of isothermal flow. The presented results identify conditions in the (Re, Ra) plane where the flow is stable as well as the minimum Ra required to produce vortices for given Re. Creation of such vortices is of interest in the design of efficient heat exchangers.

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7. REFERENCES


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