A simple FEM formulation for large deflection 2D frame analysis based on position description

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Abstract

This study presents a simple formulation to treat large deflections by the finite element method. The proposed formulation does not use the displacement concept. It considers position as the main unknown variable of the problem. The strain determination is performed directly from the proposed position concept. A non-dimensional space is created and relative curvature and fibers length are calculated for both reference and deformed configurations and used to directly compute the strain energy at general points. This procedure is very simple and presents good results, as shown in the example section. The simplicity of the resulting formulation may be considered as its main attribute. Two-dimensional frame problems are analyzed by the proposed formulation.

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1. Introduction

The analysis of structures that exhibit large deflections is of great importance in the current engineering. The increasing search for economy and optimal material application leads to the conception of very flexible structures. As a consequence, the equilibrium analysis in the non-deformed position is no more acceptable for most of applications. In this sense, a lot of studies have been developed in this engineering field. For example, the analytical solution of slender bars and their simple composition have been studied by various authors over the last years [1–12]. This approach is quite complicate and not general, as the superposition of effects is not valid for non-linear applications.

In order to create automatic, general and reliable tools for the analysis of largely deflected structures, various researchers have presented important contributions regarding finite element procedures [13–30,32–40]. These researches are very important to the development of the human knowledge of the subject, clarifying and opening the understanding of the current researchers.
One should explicitly mention the notable study by Simo et al. [40] due to the completeness of its mathematical considerations and range of applications. It seems, from general consulted references, that no other research presents such a high scientific standard on the non-linear frame analysis subject.

In the present study a simple engineering language is used to present a geometrical non-linear formulation based on position description. The position description presented here makes use of an intermediate non-dimensional space that allows defining a non-linear “engineering” strain measure calculated from a relative fiber length for different positions of the body analyzed.

The principle of minimum potential energy is applied, considering a simple linear hyper-elastic constitutive relation. Euler–Bernoulli kinematics is adopted due to its simplicity. Various examples are shown at the end of the paper, comparing the numerical results achieved with the analytical and other numerical solutions found in the literature.

2. The simple positional formulation

The principle of minimum potential energy can be written using position considerations (not displacements). Considering only the conservative elastic problem results in

\[ \Pi = U_e - P, \]  

where \( \Pi \) is the total potential energy, \( U_e \) is the strain energy and \( P \) is the potential energy of the applied forces.

Adopting linear constitutive relation for hyper-elastic materials, the strain energy can be written for the reference volume \( V_0 \) as

\[ U_e = \int_{V_0} u_e \, dV_0 = \int_{V_0} \frac{1}{2} \sigma_e \varepsilon_e \, dV_0, \]  

where \( \sigma_e \) is defined here as the “engineering stress”, i.e., the energy conjugate of the proposed “non-linear engineering strain” \( \varepsilon_e \). The strain energy is assumed to be zero in a reference position, called non-deformed. The potential energy of applied forces is written as

\[ P = \sum X, \]  

where \( X \) is the set of positions independent of each other, which may be occupied by a point of the body. It is interesting to note that the potential energy of the applied forces may not be zero in the reference configuration. The total potential energy is written as (Fig. 1)

\[ \Pi = \int_{V_0} \frac{1}{2} \sigma_e \varepsilon_e \, dV_0 - \sum X. \]  

In order to perform numerically the integral indicated in Eq. (4), it is necessary to map the geometry of the studied body (the accepted geometric approximation) and to know its relation with the strain measurement adopted. Fig. 2 gives the central line general geometry of a curve over a plane (2D frame bar).

More detailed information about the kinematics of the considered continuum is given in Fig. 3. The curve of Fig. 2 can be parameterized as a function of a non-dimensional variable \( \zeta \) (varying from 0 to 1). In this study a linear approximation for position in \( x \) direction and a cubic approximation for \( y \) direction are imposed. When it is done addition care should be taken when the curve is not a one valued function of \( x \).

Adopting approximation previously described and the nodal parameters shown in Fig. 2, one writes

\[ x = X_1 + l_1 \zeta, \]  

where \( X_1 \) and \( l_1 \) are the coordinates and length of the first node.
Fig. 1. Total potential energy written for a body in two different positions.

Fig. 2. Curve in a 2D space.

Fig. 3. Auxiliary non-dimensional space and simple mapping.
where
\[ l_x = (X_2 - X_1). \]  
(6)

Relating \( \xi \) to \( y \) by a cubic approximation, one writes
\[ y = c\xi^3 + d\xi^2 + e\xi + f \]  
(7)

with
\[ c = [\tan(\Theta_2) + \tan(\Theta_1)]l_x - 2l_y, \]  
(8a)
\[ d = 3l_x - [\tan(\Theta_2) + 2\tan(\Theta_1)]l_x, \]  
(8b)
\[ e = \tan(\Theta_1)l_x, \]  
(8c)
\[ f = Y_1, \]  
(8d)

where \( l_x = Y_2 - Y_1 \).

In this section our attention is concentrated on the simple definition of our strain measurement leaving the proof of its objectivity to the following section. As Euler–Bernoulli hypothesis is adopted, only the longitudinal strain is considered. Imagine a fiber parallel to the central line with an initial length defined by \( d_s_0 \). After deformation, its length becomes \( d_s \) and the following non-linear engineering strain is defined [41]:
\[ e_e = \frac{d_s - d_s_0}{d_s_0}. \]  
(9)

At this point the novelty is the calculation of the proposed strain measure that can be performed by relative length calculations referred to the non-dimensional space represented here by variable \( \xi \) (see Fig. 2), i.e.
\[ e_e = \frac{d_s - d_s_0}{d_s_0} = \frac{ds/d\xi - ds_0/d\xi}{ds_0/d\xi}. \]  
(10)

The values \( ds_0/d\xi \) and \( ds/d\xi \) can be considered as “auxiliary” stretches calculated regarding the non-dimensional space. In this study the physical reference configuration \( R_0 \) is a straight frame bar.

In the initial configuration for the central line passing trough the mass center of the bar one has
\[ \frac{ds_0}{d\xi} = \sqrt{\left(\frac{dx_0}{d\xi}\right)^2 + \left(\frac{dy_0}{d\xi}\right)^2} = \sqrt{(l_x^0)^2 + (l_y^0)^2} = l_0 \]  
(11)

or
\[ ds_0 = l_0 dt, \]  
(12)

where \( l_0 \) is the initial length of the finite element. A general configuration, for any instant, is described by the approximation defined in Fig. 2. For this case the central line auxiliary stretch in computed as
\[ \frac{ds}{d\xi}_{central} = \sqrt{\left(\frac{dx}{d\xi}\right)^2 + \left(\frac{dy}{d\xi}\right)^2} = \sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2} \]  
(13)

or
\[ ds_{central} = \sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2} d\xi. \]  
(14)
Applying Eq. (10), for central line one calculates

$$\varepsilon_{\text{central}} = \frac{1}{l_0} \sqrt{(l_x)^2 + (3ct^2 + 2dt + e)^2} - 1. \quad (15)$$

Following usual engineering procedures, for Euler–Bernoulli (originally straight) frame element, the strain at a fiber which is “z” distant from the central line can be written as

$$\varepsilon_e = \varepsilon_{\text{central}} + \frac{1}{r}z, \quad (16)$$

where $1/r$ is the exact curvature of the central line depicted in Fig. 2. This curvature is given as a function of $\xi$ only, as one can see in [31]:

$$\frac{1}{r} = \frac{dx \, d^2y}{d\xi \, d^2\xi} \left( \sqrt{(dy/d\xi)^2 + (dx/d\xi)^2} \right)^3. \quad (17)$$

Replacing the known approximations, i.e. Eqs. (5)–(8), Eq. (17) becomes

$$\frac{1}{r} = l_0(6ct + 2d) \left( \sqrt{l_x^2 + (3ct^2 + 2dt + e)^2} \right)^3. \quad (18)$$

Substituting Eq. (16) into Eq. (2) results in

$$U_e(\text{espec}) = \frac{E}{2} \left( \varepsilon_{\text{central}} + \frac{1}{r}z \right)^2 = \frac{1}{2} \left( \varepsilon_{\text{central}} \right)^2 + 2\varepsilon_{\text{central}} \frac{1}{r}z + \left( \frac{1}{r}z \right)^2. \quad (19)$$

In order to calculate the strain energy it is necessary to integrate the specific strain energy ($U_e(\text{espec})$) over the initial volume of the analyzed body, as the proposed strain measure is, by nature, a Lagrangian variable.

3. Objectivity of the proposed strain measure

Before describing the numerical method based on Eqs. (4) and (19) to solve geometrical non-linear problems, the objectivity of the proposed strain measure should be considered in order to allow the application of the resulting formulation to general large deflection analysis as emphasized in important papers as [40–42].

The general position of points for the analysed continuum is described by a simple mapping from a rectangular auxiliary space, see Fig. 3.

From Fig. 3 one writes the mapping, respecting Euler–Bernoulli kinematics, by the following relations:

$$X_P = X_{P_0}(\xi) - z(\eta) \frac{dY_{P_0}}{d\xi} \frac{1}{\sqrt{\left( \frac{dY_{P_0}}{d\xi} \right)^2 + \left( \frac{dX_{P_0}}{d\xi} \right)^2}}, \quad (20)$$

$$Y_P = Y_{P_0}(t) + z(\eta) \frac{dX_{P_0}}{d\xi} \frac{1}{\sqrt{\left( \frac{dY_{P_0}}{d\xi} \right)^2 + \left( \frac{dX_{P_0}}{d\xi} \right)^2}}, \quad (21)$$

where $P$ is a general point of the continuum and $P_0$ is a point of the central line of the element.

This mapping represents a deformation from the auxiliary non-dimensional space to any real position of the body, therefore the usual non-linear continuum mechanics concepts may be applied. The deformation gradient of this mapping is given by
One can calculate the stretches \((\lambda_\zeta\) and \(\lambda_\eta)\) following the reference directions ‘‘\(\zeta\)’’ and ‘‘\(\eta\)’’ defined here by the following unit vector \(M_\zeta = [1, 0]^T\) and \(M_\eta = [0, 1]^T\) as follows [41]:

\[\lambda_\zeta = \lambda(M_\zeta) = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix},\]

\[\lambda_\eta = \lambda(M_\eta) = \begin{bmatrix} A & 0 \\ 0 & 1 \end{bmatrix} = 1.\]

It is important to note that for this formulation the third direction is considered non-deformable and therefore a principal direction, i.e., \(\lambda_3 = 1\). The angle between vectors \(M_\zeta\) and \(M_\eta\), after deformation, is easily calculated, resulting:

\[\cos \theta = [1 \ 0] [A^T \ A] \begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0.\]

It means that no distortion occurs following the prescribed kinematics, therefore the calculated stretches are the principal ones, i.e., \(\lambda_\zeta = \lambda_1\) and \(\lambda_\eta = \lambda_2\). This reasoning naturally results in

\[J = \lambda_1 \lambda_2 \lambda_3 = \lambda_1.\]

As \(J\) is objective [41], the stretch

\[\lambda_1 = \frac{ds}{dt}\]

is also objective.

The proposed strain measure, expression (10), is composed of stretch (27) and (for this particular case) constant value \(ds_0/dt\), therefore one concludes that it is objective.

4. The numerical method

The technique proposed is similar to any finite element method except the determination of strain, where the auxiliary space is originally used. It is necessary to integrate the specific strain energy, Eq. (19), over the bar volume. Integrating it over the cross-section area results in

\[U_e = \frac{EA}{2}(\varepsilon_{\text{central}})^2 + \frac{EI}{2}\left(\frac{1}{r}\right)^2.\]

Now one integrates the strain energy per unit of length, Eq. (28), along the original length of the bar, i.e.,

\[U_e = \int_0^1 \frac{EA}{2}(\varepsilon_{\text{central}})^2 + \frac{EI}{2}\left(\frac{1}{r}\right)^2 l_0 d\zeta = l_0 \int_0^1 u_e d\zeta.\]
\[ \Pi = l_0 \int_0^1 u_e d\xi - F_{x1}X_1 - F_{y1}Y_1 - M_1 \Theta_1 - F_{x2}X_2 - F_{y2}Y_2 - M_2 \Theta_2, \]  
\tag{30}

where \((X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2)\) are nodal positions and \((F_{x1}, F_{y1}, M_1, F_{x2}, F_{y2}, M_2)\) are their conjugate forces. As there is no singularity in the strain energy integral one can derive Eq. (30) regarding nodal positions. The next equation shows this procedure for position \(X_1\).

\[ \frac{\partial \Pi}{\partial X_1} = l_0 \int_0^1 \frac{\partial u_e}{\partial X_1} d\xi - F_{x1} = 0. \]  
\tag{31}

The numerical strategy is to develop derivatives inside integrals and integrate them numerically regarding the non-dimensional variable \(\xi\). As it can be noted the numerical integral result is not linear regarding nodal positions. Therefore, one writes the above system of equations in the following generic way:

\[
\begin{align*}
  g_1(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_1(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - F_{x1} = 0, \\
  g_2(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_2(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - F_{y1} = 0, \\
  g_3(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_3(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - M_1 = 0, \\
  g_4(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_4(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - F_{x2} = 0, \\
  g_5(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_5(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - F_{y2} = 0, \\
  g_6(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) &= f_6(X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2) - M_2 = 0,
\end{align*}
\]  
\tag{32}

or in a compact notation:

\[
\frac{\partial \Pi}{\partial p_i} = g_i(p) = f_i(p) - F_i = 0,
\]  
\tag{33}

where \(p_i\) is a generalized parameter and indices are related to nodal positions by \((1, 2, 3, 4, 5, 6) = (X_1, Y_1, \Theta_1, X_2, Y_2, \Theta_2)\).

In a vector representation one has

\[
\begin{equation}
  g(p, F) = 0 \tag{34}
\end{equation}
\]
or

\[
\begin{equation}
  f(p) - F = 0. \tag{35}
\end{equation}
\]

It is important to observe that in this study the applied forces are not dependent on space. Space dependent forces are easily implemented if desired. The vector function \(g(p)\) is non-linear regarding nodal parameters \((p\) and \(F\)). To solve Eq. (34) one can use the Newton–Raphson procedure, i.e.

\[
\begin{equation}
  g(p) = 0 \Leftrightarrow g(p_0) + \nabla g(p_0) \Delta p \tag{36}
\end{equation}
\]
or

\[
\begin{equation}
  \Delta p = -[\nabla g(p_0)]^{-1} g(p_0), \tag{37}
\end{equation}
\]

where \(p\) is any position and \(p_0\) is the initial position.

At this point all usual words of non-linear analysis could be introduced, but the reader is invited to understand the procedure as a simple non-linear system solver. One can calculate the Hessian (of strain energy) matrix \(\nabla g(p_0)\) from expressions (30) and (33), as

\[
\begin{equation}
  \nabla g(p_0) = g_{i,k}(p_0) = f_{i,k}(p_0) - F_{i,k}, \tag{38}
\end{equation}
\]
where \( i = 1 \text{–} 6 \); \( k = 1 \text{–} 6 \) for parametric positions and \( \ell = 7 \text{–} 12 \) for forces. It is easy to achieve the following representation:

\[
\nabla g(p_0) = l_0 \int_0^1 u_{e,ik} \, dt|_{p_0} - \delta_{ij}.
\]

(39)

In order to solve Eq. (37) one needs to calculate \( g(p_0) \), i.e.

\[
g(p_0) = l_0 \int_0^1 u_{e,ik} \, dt|_{p_0} - F_i.
\]

(40)

The iterative (Newton–Raphson) process is summarized as follows:

1. Assume \( p_0 \) as the initial configuration (non-deformed). Calculate \( g(p_0) \) following Eq. (40).
2. For this \( p_0 \) calculate the Hessian matrix per unity of length, \( u_{e,ik}|_{p_0} \). Integrate this value as indicated in (39) and the result is the gradient of \( g \) at \( p_0 \).
3. Solve the system of Eq. (37) and determine \( \Delta p \)
4. Update position \( p_0 = p_0 + \Delta p \). Return to step 1 until \( \Delta p \) is sufficiently small.

Theoretically the process is not incremental. However dividing the total loading (or prescribed position) in cumulative steps helps to start the iterative procedure at a position nearer to the final desired result, reducing the number of iterations. The incremental procedure is summarized as follows:

(a) \( p_0 \) Initial position
(b) \( p_0 = p_0 + \Delta f \), where \( \Delta f \) is an increment of load or position stored in a single vector
(c) \( \{1,2,3,4\} \) iterations
(d) Return to item (b)

The algebraic steps necessary to implement the proposed kinematics and approximations are given in the electronic appendix hosted at the Internet address: http://www.set.eesc.sc.usp.br/docentes/coda/grupo/mgreco/Appendix.pdf.

5. Numerical examples

5.1. Pure bending of an Euler beam

This example is an Euler beam initially horizontal, clamped at one end and subjected to an applied moment at the other. The adopted parameters are: \( L = 500 \text{ cm} \), \( A = 20 \text{ cm}^2 \), \( E = 2 \times 10^6 \text{ N/cm}^2 \) and \( I = 2000 \text{ cm}^4 \). Six load steps and one hundred finite elements were used to run this problem until an entire lap of the geometry. The solution at each load step is obtained in seven iterations. A tolerance of \( 10^{-6} \) for absolute position residuals is adopted. A mesh of ten finite elements is also used to run the problem; more iterations are needed to reach the same accuracy achieved for one hundred finite elements.

In Fig. 4 the horizontal and vertical positions of the free node are compared with the analytical solution. It is important to note that intermediate values have been used in order to show the total path behaviour.

In Fig. 5 the deformed shapes during ‘loading’ are depicted for 10 finite elements. Again intermediate positions are shown in order to illustrate the example.

In Fig. 6 results for 10 and 100 finite elements are compared with the analytical solution for bending moment/rotation relation.
It must be noted that Simo et al. [40] solved this example using Reissner kinematics and quadratic approximation for displacement and rotation. They achieved the exact solution for each load step with only two iterations and ten finite elements.

5.2. Euler elastica

A clamped bar, Fig. 7, with length $L = 2$ m subjected to the action of an increasing compressive load from 0 to 37,100 kN is analyzed. The physical properties are: $I = 2.425 \times 10^{-5}$ m$^4$, $A = 0.0175$ m$^2$ and $E = 210 \times 10^9$ Pa. The Euler critical load is $P_{cr} = 3141.5$ kN. Ten finite elements are used to solve this problem.

In Fig. 8 a position curve of the loaded point is shown for initial eccentricities of $L/1000$, $L/100$ and $L/10$, measured at the top of the column. Only four load steps are necessary to run this problem and a tolerance of $10^{-6}$ for absolute position residuals is adopted. The initial shape is considered parabolic. In Fig. 8 the critical reference value is used to show the accuracy of results. In Fig. 9 various deformed shapes, for eccentricity $L/100$, show the pre-critical, critical and post-critical behaviors of the column.
5.3. Arch subjected to vertical load

This example shows the behavior of a parabolic arch (for example a footbridge) subjected to a central load (position control). The analysis starts from the initial position and keeps showing the post-critical behavior of the arch. The adopted geometry and physical properties are: \( L_x = 500 \text{ cm}, \ L_y = 30 \text{ cm}, \ A = 20 \text{ cm}^2, \ E = 2000 \text{ N/cm}^2 \) and \( I = 200 \text{ cm}^4 \). Shapes for important steps are given in Figs. 10–13. In Fig. 14 the position of the central point versus the actual loading is depicted and compared with results obtained using ADINA®. To run this example 100 finite elements are used without considering symmetry and a step of 0.1 N is considered to show the total path. The program is capable of running this example using a 1 N step together with the position control.

It is interesting to comment that from steps 1 to 429 the shape pattern is the one depicted in Fig. 11 and at step 400 the two ‘hills’ become the half height of the ‘valley’, when referred to zero, and the sign of the force changes. From steps 418 to 471 almost no change in load is noted and ‘hills’ turn into ‘valleys’. This is the critical region where the structure assumes indifferent positions. After step 574 the position pattern depicted in Fig. 13 is maintained.

Fig. 14 shows ADINA® results until position –10, where indifferent equilibrium is reached.
5.4. Two bar truss

This example is usual to test the large strain behavior of trusses. It consists in applying position control for a simple truss structure, Fig. 15. Only one finite element has been used due to the symmetry of the problem. In Fig. 16 a position versus load curve is depicted. The proposed formulation, using non-linear
engineering strain measure, achieves an exact solution for one iteration. Another strain measure is also used in this example. It is called here the logarithm strain measure and is given by

$$\varepsilon = \frac{ds - ds_0}{ds} = \frac{ds/d\xi - ds_0/d\xi}{ds/d\xi}.$$

The adopted physical properties are $h = a = 1$, $I = 10,000$, $E = 1$ and $A = 0.5$. 

Fig. 10. Initial position.

Fig. 11. Step 429 inside the critical zone, last step in the pre-critical shape.

Fig. 12. Step 430 inside the critical zone, first step in the post-critical shape.

Fig. 13. Step 610, the shape is almost the initial one in the reverse position.
Fig. 14. Position of the central point versus applied force (position control).

Fig. 15. Two bars truss.

Fig. 16. Position versus reaction.
As expected the logarithm strain measure provided a stiffened behavior in compression and a softened behavior in tension. In this case the same linear relation between strain and stress is adopted for both strain measures.

5.5. Pined fixed diamond frame

Jenkins et al. [11] solved this example analytically, using elliptic integrals. Mattiasson [2] performed the numerical evaluation of the elliptic integrals providing a valuable test for finite element formulations of large deflections. The following properties are adopted to run the problem: \( L = 1, \ E = 1, \ I = 1 \) and \( A = 1000 \). Fig. 17 shows the diamond frames and the measured “displacements”. Displacements are calculated here by the difference between positions to allow comparisons with Ref. [2]. Symmetry is adopted.

Twenty finite elements were used to run this problem. In Figs. 18 and 19 the achieved displacements are depicted together with the analytical solution calculated by Mattiasson [2]. The maximum achieved errors

Fig. 17. Diamond frame, static scheme for tension \((P > 0)\) and compression \((P < 0)\) situations.

Fig. 18. Horizontal displacement versus load for diamond frame.
Fig. 19. Vertical displacement versus load for diamond frame.

Fig. 20. Compression diamond frame; deformed shapes.

Fig. 21. Tension diamond frame; deformed shape.
are 1% in tension and 0.3% in compression. The adopted tolerance is $Tol = 10^{-5}$ in position. In Figs. 20 and 21 some deformed shapes are depicted.

6. Conclusions

In this paper a consistent and simple formulation is proposed to solve geometrically non-linear plane frame problems. It is based on position description and calculates strains from relative lengths and curvatures using an intermediate non-dimensional space. From this procedure an engineering non-linear strain measure is presented. In order to show the didactic possibilities of the technique, a simple engineering language is used. The formulation developed can be used for practical applications and presents good convergence and accuracy.

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