INTERNAL FORCE EVALUATION FOR REISSNER-MINDLIN PLATES USING THE BOUNDARY ELEMENT METHOD

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ABSTRACT

In this article, the accuracy of computing internal forces in BEM formulation for Reissner’s and Mindlin Plates is discussed. An accurate scheme to evaluate the initial moment domain integral usually used to compute the vector correction in non-linear analysis is proposed. The domain integrals are transformed to boundaries the of the approximation sub-domains, resulting into regular integrals, which are accurately performed by mean of standard integration schemes.

Keywords: Boundary elements, plate bending problems.

1 INTRODUCTION

The direct boundary element formulation for Reissner’s and Mindlin’s plates were proposed by WEEÉN (1982) and BARCELLOS & SILVA (1987), respectively. Recently, these formulations have been discussed in several works, among them we wish to point out the unified BEM approaches, proposed separately by WESTPHAL et al. (2001) and PALERMO (2003).

As these formulations deal with complex kernels, one must be sure that the integrals along boundary elements and over internal cells are accurately evaluated. Studies regarding this matter have been presented in several works as in: RASHED et al. (1998), EL-ZAFRANY et al. (1995) and MARCZAK & CREUS (2002). All these works are related with the accuracy for computing boundary element integrals. In this paper, we are trying to improve the accuracy to evaluate the initial moment field effects required to perform non-linear analysis. A simple scheme to evaluate the initial moment effects over triangular cells with linear approximations is proposed. From the unified approach given by PALERMO (2003) the integrals appearing in both displacement and internal force equations are easily transformed to the boundary and them precisely computed, leading to very accurate values of deflections, and bending and twisting moments. Simple examples are solved to show the accuracy reached by using the proposed formulation.

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2 INTEGRAL REPRESENTATIONS OF DEFLECTIONS AND INTERNAL FORCES

Let us consider a plate of constant thickness $h$ described in the domain $\Omega$ with boundary $\Gamma$, over which the in-plane Cartesian axes are $x_1, x_2$, being $x_3$ orthogonal to the plate middle surface.

The generalized integral representation of displacements obtained by using a weighted residual statement or the Betti’s reciprocal work is given by\(^3,8\):

$$
c_{ik}u_k = \int_{\Gamma}^{} u_{ik}^* p_i d\Gamma - \int_{\Gamma}^{} p_{ik}^* u_k d\Gamma + \int_{\Omega}^{} \left[ u_{i3}^* - M u_{i\alpha\tau^*} \right] g d\Omega + \int_{\Omega}^{} u_{i\alpha}^* \beta^0_{\alpha\beta} d\Omega
$$

(1)

where $u_k$ and $p_k$ are generalized displacement and traction components, respectively, $g$ is distributed load orthogonally applied on the plate surface, the values marked by * are the well-known fundamental solutions, $m_{\alpha\beta}^0$ represents an applied initial moment field, $u_{i\alpha}^* \beta$ gives curvatures and $c_{ik}$ is the free terms depending on the position of the load point.

The parameter $M$ is zero for Mindlin’s plate and $\nu/(1-\nu)\lambda^2$ for the Reissner’s plates, being $\lambda = \sqrt{10}/h$ a constant related with the shear effects.

According to PALERMO (2003), the fundamental solution of the plate problem for Kirchhoff, Reissner and Mindlin models, i.e., rotations and deflections due to a unit point or unit couple are given by:

- Due to a unit point load in the direction orthogonal to the plate surface:

$$
\phi_a = -\frac{r^2}{8\pi D} r^2 (ln \lambda r - 1) r_{\alpha a}
$$

(2)

$$
w^*(s,q) = \frac{1}{\pi D} \left[ \frac{L}{8} r^2 (ln \lambda r - 1) \right] - \frac{\ell n \lambda r}{(1-\nu)\lambda^2}
$$

(3)

- Due to a unit couple in the direction $x_{\alpha}$:

$$
u_{i\alpha}^* = -w_{i\alpha} + \delta_{i\alpha} h_{\xi\xi} - h_{i\alpha}
$$

(4)

and

$$
u_{i\alpha}^* = -w_{i\alpha}
$$

(5)
where

$$w_{\alpha \beta} = \frac{1}{8\pi D} \frac{\partial}{\partial x_\alpha} \left[ r^2 \left( \ln r - 1 \right) \right]$$

(6)

$$h_{\alpha \beta} = \frac{1}{\pi D (1 - \nu) \lambda^2} \frac{\partial}{\partial x_\beta} \left[ \ln (\lambda r) + K_0 (\lambda r) \right]$$

(7)

$$w_{\alpha} = \frac{1}{8\pi D} \frac{\partial}{\partial x_\alpha} \left[ r^2 (\ln r - 1) \right]$$

(8)

where $K_0$ is the modified Bessel function.

Let us now work on the initial moment term, appearing in equation (1). Before approximating the initial moment values over sub-domain or cells, we can reduce its singularity, by integration the corresponding domain term by parts to give:

$$\int_{\Omega} \alpha \beta \beta \alpha \Omega \sum_{\Gamma} \int \Gamma \alpha \beta \beta \alpha \Omega \Gamma \eta \delta \eta \eta \Gamma \eta \delta \eta \delta \Gamma \delta \Omega$$

(9)

where $\sum \Gamma_m$ is the external boundary $\Gamma$ if $m^0_{\alpha \beta}$ were continuous inside the body, otherwise the total sub-domain (cell) boundaries.

We can use the fundamental solution decomposition given in equation (4) to modify the integrals in equation (9) that after integrating by parts the remaining domain integral becomes:

$$\int_{\Omega} \alpha \beta \beta \alpha \Omega \sum_{\Gamma_m} \int \Gamma_m \alpha \beta \beta \alpha \Omega \Gamma \eta \delta \eta \eta \Gamma \eta \delta \eta \delta \Gamma \delta \Omega$$

(10)

It is worth to stress that the integrals with density $m^0_{\alpha \beta \gamma}$ is usually performed along the external boundary only, while the integrals with density $m^0_{\alpha \beta \gamma \eta}$ have to be also performed along the cell boundary as continuity of $m^0_{\alpha \beta \gamma \eta}$ is usually not assumed. Moreover, the domain integrals were all eliminated because only linear shape
functions (over continuous and discontinuous internal cells) will be used to approach \( m_{\alpha\beta} \).

The domain integral due to the initial moment field was therefore transformed to the boundary or sub-domain boundaries, along which no strong singularity has to be evaluated. Thus, simple numerical integration techniques can be adopted to evaluate accurately the initial moment effects.

Similarly, from the displacement representation we can derive curvature representations and them internal moment and shear force integral equations. For simplicity, let us particularize the problem deriving only bending and twisting moment representations, which are given by:

\[
\begin{align*}
\Gamma_{\alpha\beta}^\omega & = \int \int \int \left[ \frac{\partial}{\partial x_\beta} \int u_{a\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega + \frac{\partial}{\partial x_\alpha} \int u_{\beta\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega + \frac{2\nu}{1-\nu} \frac{\partial}{\partial x_\eta} \int u_{\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega \right] d\Omega - m_{\alpha\beta}^\omega + \\
& + D(1-\nu) \sum_m \left[ \int \left( u_{a\gamma,\gamma}^\omega + u_{\beta\gamma,\gamma}^\omega + \frac{2\nu}{1-\nu} \delta_{\alpha\gamma} u_{\gamma,\gamma}^\omega \right) \eta_{\beta} m_{\alpha\beta}^\omega d\Gamma \right] \\
& + D(1-\nu) \sum_m \left[ - \int \left( u_{a\gamma,\delta}^\omega + u_{\beta\gamma,\delta}^\omega + \frac{2\nu}{1-\nu} \delta_{\alpha\gamma} u_{\gamma,\delta}^\omega \right) \eta_{\beta} m_{\alpha\beta}^\omega d\Gamma \right]
\end{align*}
\] (11)

The integrals along the boundary can be accurately performed using analytical or appropriate numerical schemes as has been shown elsewhere\(^5^6\). Herein, we are going to work on the integral terms containing \( m_{\alpha\beta}^\omega \). By integrating these terms by parts and then differencing them we find:

\[
\begin{align*}
\Gamma_{\alpha\beta}^\omega & = \int \int \int \left[ \frac{\partial}{\partial x_\beta} \int u_{a\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega + \frac{\partial}{\partial x_\alpha} \int u_{\beta\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega + \frac{2\nu}{1-\nu} \frac{\partial}{\partial x_\eta} \int u_{\gamma,\gamma}^\omega m_{\alpha\beta}^\omega d\Omega \right] d\Omega - m_{\alpha\beta}^\omega + \\
& + D(1-\nu) \sum_m \left[ \int \left( u_{a\gamma,\gamma}^\omega + u_{\beta\gamma,\gamma}^\omega + \frac{2\nu}{1-\nu} \delta_{\alpha\gamma} u_{\gamma,\gamma}^\omega \right) \eta_{\beta} m_{\alpha\beta}^\omega d\Gamma \right] \\
& + D(1-\nu) \sum_m \left[ - \int \left( u_{a\gamma,\delta}^\omega + u_{\beta\gamma,\delta}^\omega + \frac{2\nu}{1-\nu} \delta_{\alpha\gamma} u_{\gamma,\delta}^\omega \right) \eta_{\beta} m_{\alpha\beta}^\omega d\Gamma \right]
\end{align*}
\] (12)

The fundamental solutions in equation (12) can be conveniently replaced by the functions given in equations (4). As already described for equation (10), no remaining integral is left when \( m_{\alpha\beta}^\omega \) is linearly approximated.

The resulting integrals containing the densities \( m_{\alpha\beta}^\omega \) and \( m_{\alpha\beta,\gamma}^\omega \) are not singular and can be performed analytically or numerically along the sub-region (cells) boundaries \( \Gamma_m \) without requiring any special scheme. To perform the numerical tests for the next section, we have used the classical Gauss scheme with sub-elements.
3 NUMERICAL TESTS

To demonstrate the accuracy for computing the domain integrals involving the initial moment $m_{\alpha\beta}^0$, we selected two simple problems with known exact solution: constant and linear moment fields applied over rectangular domains.

Let us first consider a rectangular domain $\ell \times b$ ($\ell$ in the $x_1$ direction), over which a constant initial moment field, $m_{ij}^0 = 1.0$, is applied. The plate is simply supported along the sides of length $b$ and free in the other direction. The example was analysed using a very poor mesh and a finer one. As the exact solution is quadratic in $w$, the same order of the approximations, practically no variation is observed refining the mesh.

Exact values of deflections and internal moments were computed all over the plate domain. In Table 1, we depicted the computed values along the plate middle axis $x_1$. Bending and twisting moments are zero, therefore the corresponding obtained values are exactly the computed errors, demonstrating that the technique is rather accurate. Numerical and exact values for deflections are also given in Table 1, confirming again the accuracy of the formulation. The maximum error verified for computing deflections is also of order of $10^{-8}$, the same errors observed for computing bending and twisting moments.

The second test carried out consists of applying a linear initial moment field varying from zero at $x_1=0.0$ to $1.00$ at $x_1=\ell$, i.e., $m_{ii}^0 = x_1/\ell$. As we are using quadratic approximations to approach all boundary values, only approximate answers were expected. Table 2 gives the results obtained by using only the finer mesh. Deflections and moments, $m_{ii}$, were computed for points along $x_2$, the axis passing through the plate centre in the side $b$ direction. It should be noted that the exact moments are zero, therefore the values in Table 2 represent again the computed errors.
Table 2. Deflections and moments due to $m_{i1}^0 = x_1 / \ell$

<table>
<thead>
<tr>
<th>$x_2 / b$</th>
<th>0.005</th>
<th>0.0125</th>
<th>0.5</th>
</tr>
</thead>
<tbody>
<tr>
<td>Solution</td>
<td>Exact</td>
<td>Num.</td>
<td>Exact</td>
</tr>
<tr>
<td>$m_{i1} \ell^2 / D$</td>
<td>0.0</td>
<td>02.33E-2</td>
<td>0.0</td>
</tr>
<tr>
<td>$\omega D / (m_{i1}^0 \ell^2)$</td>
<td>0.0625</td>
<td>0.0625</td>
<td>0.0625</td>
</tr>
</tbody>
</table>

The deflections are precisely computed everywhere, being the accuracy excellent (error of order of $10^{-8}$). On the contrary, moments near the boundary are affected by the approximation of the boundary values. Node defined to close to the boundary lead to inaccurate numerical bending and twisting moment. For the node $x_2 / b = 0.005$, Table 2, whose ratio between its distance to boundary and the element length is $d / \ell = 0.04$, a rather bad value was computed, whereas for points with $d / \ell = 0.1$ the moment accuracy was very good (error of $4.0 \times 10^{-6}$). Thus, it is recommended that only nodes not so close to the boundary be adopted to describe the initial moment field. The least recommended distance to define the moment field should be 0.1 of the element size inside the domain.

4 CONCLUSION

A simple and efficient domain integral transform is proposed to integrate the initial moment term for non-linear plate analysis. The transform was easily performed by using a unified approach where the fundamental solutions are given by primitive functions. The results obtained when solving very simple tests confirmed the accuracy. Thus, for more complex non-linear analysis the formulation is expected to be stable.

5 REFERENCES


PALERMO JR., L. Plate bending analysis using the classical or the Reissner-Mindlin models. *Engineering Analysis with Boundary Elements*, 2003 (to appear)


